

A PRODUCT OF TENSOR PRODUCT L -FUNCTIONS OF QUASI-SPLIT CLASSICAL GROUPS OF HERMITIAN TYPE

DIHUA JIANG AND LEI ZHANG

ABSTRACT. A family of global integrals representing a product of tensor product (partial) L -functions:

$$L^S(s, \pi \times \tau_1) L^S(s, \pi \times \tau_2) \cdots L^S(s, \pi \times \tau_r)$$

are established in this paper, where π is an irreducible cuspidal automorphic representation of a quasi-split classical group of Hermitian type and τ_1, \dots, τ_r are irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_{a_1}, \dots, \mathrm{GL}_{a_r}$, respectively. When $r = 1$ and the classical group is an orthogonal group, this was studied by Ginzburg, Piatetski-Shapiro and Rallis in 1997 and when π is generic and τ_1, \dots, τ_r are not isomorphic to each other, this is considered by Ginzburg, Rallis and Soudry in 2011. In this paper, we prove that the global integrals are eulerian and finish the explicit calculation of unramified local L -factors in general. The remaining local and global theory for this family of global integrals will be considered in our future work.

1. INTRODUCTION

Let F be a number field and E be a quadratic extension of F when we discuss unitary groups and E be equal to F when we discuss orthogonal groups. Let G_n be a quasi-split group, which is either $\mathrm{U}_{n,n}$, $\mathrm{U}_{n+1,n}$, SO_{2n+1} , or SO_{2n} , defined over F . Let \mathbb{A}_E be the ring of adeles of E and \mathbb{A} be the ring of adeles of F . Take τ to be an irreducible generic automorphic representation of $\mathrm{Res}_{E/F}(\mathrm{GL}_a)(\mathbb{A}) = \mathrm{GL}_a(\mathbb{A}_E)$ of isobaric type, i.e.

$$(1.1) \quad \tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r$$

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where $a = \sum_{i=1}^r a_i$ is a partition of a and τ_i is an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{a_i}(\mathbb{A}_E)$. Let π be an irreducible cuspidal automorphic representation of $G_n(\mathbb{A})$. We consider a family of global zeta integrals (see Section 3 for definition), which represent the family of the tensor product (partial) L -functions $L^S(s, \pi \times \tau)$, which is expressed as follows:

$$(1.2) \quad L^S(s, \pi \times \tau) = L^S(s, \pi \times \tau_1) L^S(s, \pi \times \tau_2) \cdots L^S(s, \pi \times \tau_r).$$

It is often interesting and important in Number Theory and Arithmetic to consider certain simultaneous behavior at a certain given point $s = s_0$. For instance, the nonvanishing at $s = \frac{1}{2}$, the center of the symmetry of the L -functions $L^S(s, \pi \times \tau_1)$, $L^S(s, \pi \times \tau_2)$, \cdots , $L^S(s, \pi \times \tau_r)$, or particularly, taking $\tau_1 = \tau_2 = \cdots = \tau_r$, which yields the r -th power $L^S(s, \pi \times \tau_1)^r$ for all positive integers r . As remarked at the end of this paper, the arguments and the methods still work if one replaces the single variable s by multi-variable (s_1, \cdots, s_r) . However, we focus on the case of single variable s in this paper.

We use a family of the Bessel periods (discussed in Section 2) to define the family of global zeta integrals, following closely the formulation of Ginzburg, Piatetski-Shapiro and Rallis in [GPSR97], where the case when $r = 1$ and G_n is an orthogonal group was considered. When π is generic, i.e. has a nonzero Whittaker-Fourier coefficient, and τ_1, \cdots, τ_r are not isomorphic to each other, this family of tensor product L -functions were studied by Ginzburg, Rallis and Soudry in their recent book [GRS11]. However, the global integrals studied in [GRS11] can not cover the general situation considered in this paper, while the global zeta integrals here are the most general version of this kind stated from the pioneer work of Piatetski-Shapiro and Rallis and of Gelbart and Piatetski-Shapiro ([GPSR87]). Some more special cases were studied earlier by various authors and we refer to the relevant discussions in [GPSR97] and [GRS11].

In addition to the potential application towards the simultaneous nonvanishing of the central values of the tensor product L -functions, the basic relation between the product of the tensor product (partial) L -functions and the family of global zeta integrals is also an important ingredient in the proof of the nonvanishing of the certain explicit constructions of endoscopy correspondences as indicated for some special cases in the work of Ginzburg in [G08], and as generally formulated in the work of the first name author in ([J11] and [J12]). We will come back to this topic in our future work ([JZ13]).

In general, the meromorphic continuation to the whole complex plane of the product of the tensor product (partial) L -functions is known

from the work of R. Langlands on the explicit calculation of the constant terms of Eisenstein series ([L71]). However, when π is not generic, i.e. has no nonzero Whittaker-Fourier coefficients, the Langlands conjecture on the standard functional equation and the finite number of poles for $\operatorname{Re}(s) \geq \frac{1}{2}$ is still not known ([Sh10]).

According to the recent work of Arthur ([Ar12]) and also of C.-P. Mok ([Mk12]), when π has a generic global Arthur parameter, the tensor product (complete) L -functions $L(s, \pi \times \tau)$ can be defined through the Arthur-Langlands transfer from G_n to general linear groups. In this case, π is nearly equivalent to an irreducible generic cuspidal automorphic representation π_0 of $G_n(\mathbb{A})$ and hence we have an identity for partial L -functions

$$L^S(s, \pi \times \tau) = L^S(s, \pi_0 \times \tau).$$

Hence it is an interesting problem to develop the local theory of the family of global zeta integrals considered in this paper and prove that the local L -factors produced by means of this current approach are the same as what Arthur defines in [Ar12] through the Arthur-Langlands transfers. It seems still an issue to prove that the Arthur local L -factors are compatible with the classification theory of representations of G_n over local fields. It is our hope that the complete local theory developed through the approach considered in this paper will be able to achieve this goal for general π , which may have a non-generic global Arthur parameter. This will turn out to be one of the important steps towards the completion of the local Langlands conjecture for classical groups of Hermitian type over p -adic local fields.

For groups of skew-Hermitian type, some preliminary work has been done in [GJRS11], using Fourier-Jacobi periods. Further work is in progress, including the work of X. Shen in his PhD thesis in University of Minnesota, 2013, which has produced two preprints [Sn12] and [Sn13]. A parallel theory for this case will also be considered in future.

In Section 2, we introduce a family of Eisenstein series, which is needed for the construction of the family of global zeta integrals, and discuss the family of Bessel periods which are needed to formulate the family of global zeta integrals. In Section 3, we unfold the global zeta integrals and prove that they are eulerian. In Section 4, we do the explicit calculations for local zeta integrals with unramified data to produce the unramified local L -factors of the tensor product type, following the argument of [GPSR97]. The main global result is Theorem 4.12, which is started at the end of Section 4. The ideas and methods used in the proofs in this paper will be described with more details in each section, which are essentially the extension of those used in

[GPSR97] for the orthogonal group G_n and with $r = 1$ to the generality considered in this paper.

2. CERTAIN EISENSTEIN SERIES AND BESSEL PERIODS

We introduce a family of Eisenstein series which will be used in the definition of a family of global zeta integrals, representing the family of the product of the tensor product L -functions as discussed in the introduction. The global zeta integrals are basically a family of Bessel periods of those Eisenstein series. We recall first the general notion of the Bessel periods of automorphic forms from [GPSR97], [GJR09], [BS09] and [GRS11].

Let F be a number field. Define $E = F$ or $E = F(\sqrt{\rho})$, a quadratic extension of F , depending on that the classical group we considered is orthogonal or unitary, accordingly. It follows that the Galois group of E/F is either trivial or generated by non-trivial automorphism $x \mapsto \bar{x}$. The ring of adeles of F is denoted by \mathbb{A} , while the ring of adeles of E is denoted by \mathbb{A}_E .

Let V be a E -vector space of dimension m with a non-degenerate quadratic form q_V if $E = F$ or a non-degenerate Hermitian form (also denoted by q_V) if $E = F(\sqrt{\rho})$. Let $U(q_V)$ be the connected component of isometry group of (V, q_V) defined over F . It follows that $U(q_V)$ is a special orthogonal group or a unitary group. Let $\tilde{m} = \text{Witt}(V)$ be the Witt index of V . Let V^+ be a maximal totally isotropic subspace of V and V^- be its dual, so that V has the following polar decomposition

$$V = V^+ \oplus V_0 \oplus V^-,$$

where $V_0 = (V^+ \oplus V^-)^\perp$ denotes the anisotropic kernel of V . We choose a basis $\{e_1, e_2, \dots, e_{\tilde{m}}\}$ of V^+ and a basis $\{e_{-1}, e_{-2}, \dots, e_{-\tilde{m}}\}$ of V^- such that $q_V(e_i, e_{-j}) = \delta_{i,j}$ for all $1 \leq i, j \leq \tilde{m}$.

We assume in this paper that the algebraic F -group $U(q_V)$ is F -quasi-split. Then the anisotropic kernel V_0 is at most two dimensional. More precisely, when $E = F$, if $\dim_E V = m$ is even, then $\dim_E V_0$ is either 0 or 2, and if $\dim_E V = m$ is odd, then $\dim_E V_0$ is 1; and when $E = F(\sqrt{\rho})$, $\dim_E V_0$ is 0 or 1 according to that $\dim_E V = m$ is even or odd.

When $\dim V_0 = 2$, we choose an orthogonal basis $\{e_0^{(1)}, e_0^{(2)}\}$ of V_0 with the property that

$$q_{V_0}(e_0^{(1)}, e_0^{(1)}) = 1, \quad q_{V_0}(e_0^{(2)}, e_0^{(2)}) = -c,$$

where $c \in F^\times$ is not a square and $q_{V_0} = q_V|_{V_0}$. When $\dim V_0 = 1$, we choose an anisotropic basis vector e_0 for V_0 . We put the basis in the

following order

$$(2.1) \quad e_1, e_2, \dots, e_{\tilde{m}}, e_0^{(1)}, e_0^{(2)}, e_{-\tilde{m}}, \dots, e_{-2}, e_{-1}$$

if $\dim_E V_0 = 2$;

$$(2.2) \quad e_1, e_2, \dots, e_{\tilde{m}}, e_0, e_{-\tilde{m}}, \dots, e_{-2}, e_{-1}$$

if $\dim_E V_0 = 1$; and

$$(2.3) \quad e_1, e_2, \dots, e_{\tilde{m}}, e_{-\tilde{m}}, \dots, e_{-2}, e_{-1}$$

if $\dim_E V_0 = 0$.

With the choice of the ordering of the basis vectors, the F -rational points $U(q_V)(F)$ of the algebraic group $U(q_V)$ are realized as an algebraic subgroup of $GL_m(E)$. Define $n = \lfloor \frac{m}{2} \rfloor$ and put $G_n = U(q_V)$, which is the same as given in the introduction. From now on, for any F -algebraic subgroup H of G_n , the F -rational points $H(F)$ of H are regarded as a subgroup of $GL_m(E)$. Similarly, the \mathbb{A} -rational points $H(\mathbb{A})$ of H are regarded as a subgroup of $GL_m(\mathbb{A}_E)$.

The corresponding standard flag of V (with respect to the given ordering of the basis vectors) defines an F -Borel subgroup B . We may write $B = TN$ with T a maximal F -torus, whose elements are diagonal matrices, and with N the unipotent radical of B , whose elements are upper-triangular matrices. Let T_0 be the maximal F -split torus of G_n contained in T . We define the root system $\Phi(T_0, G_n)$ with the set of positive roots $\Phi^+(T_0, G_n)$ corresponding to the Borel subgroup given above.

Let V_ℓ^+ be the totally isotropic subspace generated by $\{e_1, e_2, \dots, e_\ell\}$ and $P_\ell = M_\ell U_\ell$ be a standard maximal parabolic subgroup of G_n , which stabilizes V_ℓ^+ . The Levi subgroup M_ℓ is isomorphic to $GL(V_\ell^+) \times G_{n-\ell}$. Here $GL(V_\ell^+) = \text{Res}_{E/F}(GL_\ell)$ and $G_{n-\ell} = U(q_{W_\ell})$ with $q_{W_\ell} = q_V|_{W_\ell}$ and $W_\ell = (V_\ell^+ \oplus V_\ell^-)^\perp$.

Let $\underline{\ell} := [\ell_1 \ell_2 \dots \ell_p]$ be a partition of ℓ . Then $P_{\underline{\ell}} = M_{\underline{\ell}} U_{\underline{\ell}}$ is a standard parabolic subgroup of $\text{Res}_{E/F} GL_\ell$, whose Levi subgroup

$$M_{\underline{\ell}} \cong \text{Res}_{E/F} GL_{\ell_1} \times \text{Res}_{E/F} GL_{\ell_2} \times \dots \times \text{Res}_{E/F} GL_{\ell_p}.$$

2.1. Bessel periods. Define N_ℓ to be the unipotent subgroup of G_n consisting of elements of following type,

$$(2.4) \quad N_\ell = \left\{ n = \begin{pmatrix} z & y & x \\ & I_{m-2\ell} & y' \\ & & z^* \end{pmatrix} \in G_n \mid z \in Z_\ell \right\},$$

where Z_ℓ is the standard maximal (upper-triangular) unipotent subgroup of $\text{Res}_{E/F} GL_\ell$. It is clear that $N_\ell = U_{[1^\ell]}$ where $[1^\ell]$ is the partition of ℓ with 1 repeated ℓ times.

Fix a nontrivial character ψ_0 of $F \backslash \mathbb{A}_F$ and define a character ψ of $E \backslash \mathbb{A}_E$ by

$$(2.5) \quad \psi(x) := \begin{cases} \psi_0(x) & \text{if } E = F; \\ \psi_0(\frac{1}{2} \text{tr}_{E/F}(\frac{x}{\sqrt{\rho}})) & \text{if } E = F(\sqrt{\rho}). \end{cases}$$

Then take w_0 to be an anisotropic vector in W_ℓ and define a character ψ_{ℓ, w_0} of N_ℓ by

$$(2.6) \quad \psi_{\ell, w_0}(n) := \psi\left(\sum_{i=1}^{\ell-1} z_{i, i+1} + q_{W_\ell}(y_\ell, w_0)\right),$$

where y_ℓ is the last row of y in $n \in N_\ell$ as defined in (2.4), which is regarded as a vector in W_ℓ .

If $\ell = \tilde{m}$, ψ_{ℓ, w_0} is a generic character on the maximal unipotent group $N = N_{\tilde{m}}$. We will not consider this case here and hence we always assume that $\ell < \tilde{m}$ from now on.

For $\kappa \in F^\times$, we choose

$$(2.7) \quad w_0 = y_\kappa = e_{\tilde{m}} + (-1)^{m+1} \frac{\kappa}{2} e_{-\tilde{m}},$$

which implies that $q(y_\kappa, y_\kappa) = (-1)^{m+1} \kappa$ and that the corresponding character is

$$(2.8) \quad \psi_{\ell, \kappa}(n) = \psi_{\ell, w_0}(n) = \psi\left(\sum_{i=1}^{\ell-1} z_{i, i+1} + y_{\ell, \tilde{m}-\ell} + (-1)^{m+1} \frac{\kappa}{2} y_{\ell, m-\tilde{m}-\ell+1}\right).$$

The Levi subgroup $M_{[1^\ell]} = (\text{Res}_{E/F} \text{GL}_1)^{\times \ell} \times G_{n-\ell}$ normalizes the unipotent subgroup N_ℓ by the adjoint action, and acts on the set of the characters $\psi_{\ell, \kappa}$, with $\kappa \in F^\times$, of $N_\ell(F)$. The $M_{[1^\ell]}(F)$ -orbits are classified by the Witt Theorem and give all F -generic characters of $N_\ell(F)$. The stabilizer of ψ_{ℓ, w_0} in the Levi subgroup $M_{[1^\ell]}$ is the subgroup

$$(2.9) \quad L_{\ell, w_0} = \left\{ \begin{pmatrix} I_\ell & & \\ & \gamma & \\ & & I_\ell \end{pmatrix} \in G_n \mid \gamma J_{m-2\ell} w_0 = J_{m-2\ell} w_0 \right\} \cong H_{n-\ell},$$

where $H_{n-\ell}$ is defined to be $\text{U}(q_{W_\ell \cap w_0^\perp})$ with $q_{W_\ell \cap w_0^\perp} = q_V|_{W_\ell \cap w_0^\perp}$, and J_k is the $k \times k$ matrix defined inductively by $J_k = \begin{pmatrix} & 1 \\ J_{k-1} & \end{pmatrix}$ and $J_1 = 1$. Define

$$(2.10) \quad R_{\ell, w_0} := H_{n-\ell} N_\ell = \text{U}(q_{W_\ell \cap w_0^\perp}) N_\ell.$$

Note that $\dim_E V$ and $\dim_E W_\ell \cap w_0^\perp$ have the different parity. If $\ell = 0$, the unipotent subgroup N_0 is trivial and we have that

$$R_{0,w_0} = U(q_{V \cap w_0^\perp}).$$

When taking $w_0 = y_\kappa$, we will use the notation $\psi_{\ell,y_\kappa} = \psi_{\ell,\kappa}$, $L_{\ell,y_\kappa} = L_{\ell,\kappa}$ and $R_{\ell,y_\kappa} = R_{\ell,\kappa}$, respectively.

Let ϕ be an automorphic form on $G_n(\mathbb{A})$. Define the **Bessel-Fourier coefficient** (or Gelfand-Graev model) of ϕ by

$$(2.11) \quad \mathcal{B}^{\psi_{\ell,w_0}}(\phi)(h) := \int_{N_\ell(F) \backslash N_\ell(\mathbb{A})} \phi(nh) \psi_{\ell,w_0}^{-1}(n) \, dn.$$

This defines an automorphic function on the stabilizer $L_{\ell,w_0}(\mathbb{A}) = H_{n-\ell}(\mathbb{A})$. Take a cuspidal automorphic form φ on $H_{n-\ell}(\mathbb{A})$ and define the $(\psi_{\ell,w_0}, \varphi)$ -**Bessel period** or simply **Bessel period** of ϕ by

$$(2.12) \quad \mathcal{P}^{\psi_{\ell,w_0}}(\phi, \varphi) := \int_{H_{n-\ell}(F) \backslash H_{n-\ell}(\mathbb{A})} \mathcal{B}^{\psi_{\ell,w_0}}(\phi)(h) \varphi(h) \, dh.$$

We will apply this Bessel period to a family of Eisenstein series next.

2.2. Eisenstein series. We follow the notation of [MW95] to define a family of Eisenstein series. Let $P_j = M_j U_j$ be a standard maximal parabolic F -subgroup of G_n with the Levi subgroup

$$M_j = \text{Res}_{E/F}(\text{GL}_j) \times G_{n-j},$$

for some j with $1 \leq j \leq \tilde{m}$. When $j = \tilde{m}$, the group $G_{n-\tilde{m}}$ disappears, if $\dim_E V_0 = 0$, or $\dim_E V_0 = 1$ and $E = F$. Following [MW95, Page 5], the space X_{M_j} of all continuous homomorphisms from $M_j(\mathbb{A})$ to \mathbb{C}^\times , which is trivial on $M_j(\mathbb{A})^1$, can be identified with \mathbb{C} by the mapping $\lambda_s \leftrightarrow s$, which is normalized as in [Sh10].

Let τ be an irreducible unitary generic automorphic representation of $\text{GL}_j(\mathbb{A}_E)$ of the following isobaric type:

$$(2.13) \quad \tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r,$$

where $\underline{j} = [j_1 j_2 \cdots j_r]$ is a partition of j and τ_i is an irreducible unitary cuspidal automorphic representation of $\text{GL}_{j_i}(\mathbb{A}_E)$. Let σ be an irreducible automorphic representation of $G_{n-j}(\mathbb{A})$, which may not be cuspidal. Note that σ is irrelevant if $j = \tilde{m}$ and the group $G_{n-\tilde{m}}$ disappears. Following the definition of automorphic forms in [MW95, I.2.17], take an automorphic form

$$(2.14) \quad \phi = \phi_{\tau \otimes \sigma} \in \mathcal{A}(U_j(\mathbb{A}) M_j(F) \backslash G_n(\mathbb{A}))_{\tau \otimes \sigma}.$$

For $\lambda_s \in X_{M_j}$, the Eisenstein series associated to $\phi(g)$ is defined by

$$(2.15) \quad E(\phi, s)(g) = E(\phi_{\tau \otimes \sigma}, \lambda_s)(g) = \sum_{\delta \in P_j(F) \backslash G_n(F)} \lambda_s \phi(\delta g).$$

It is absolutely convergent for $\operatorname{Re}(s)$ large and uniformly converges for g over any compact subset of $G_n(\mathbb{A})$, has meromorphic continuation to $s \in \mathbb{C}$ and satisfies the standard functional equation.

Recall that $H_{n-\ell}$ is defined to be $U(q_{W_\ell \cap w_0^\perp})$ and that $\dim_E V$ and $\dim_E W_\ell \cap w_0^\perp$ have the different parity. Let π be an irreducible *cuspidal* automorphic representation of $H_{n-\ell}(\mathbb{A})$ and take a cuspidal automorphic form

$$(2.16) \quad \varphi \in \mathcal{A}_0(H_{n-\ell}(F) \backslash H_{n-\ell}(\mathbb{A}))_\pi.$$

The global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ is defined to be the following Bessel period

$$(2.17) \quad \mathcal{Z}(s, \varphi_\pi, \phi_{\tau \otimes \sigma}, \psi_{\ell, w_0}) := \mathcal{P}^{\psi_{\ell, w_0}}(E(\phi_{\tau \otimes \sigma}, s), \varphi_\pi).$$

Because φ_π is cuspidal, it is easy to see that the following holds.

Proposition 2.1. *The global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ converges absolutely at any $s \in \mathbb{C}$ when the Eisenstein series $E(\phi_{\tau \otimes \sigma}, s)$ has no pole at s and hence is holomorphic, and has possible poles at the locations where the Eisenstein series has poles.*

3. THE EULERIAN PROPERTY OF THE GLOBAL INTEGRALS

We prove here that the global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ will be expressed as an eulerian product of local zeta integrals. When $j = n = \lfloor \frac{m}{2} \rfloor$, such global zeta integrals with generic π have been studied in [GRS11, Chapter 10]. Hence we assume from now on that $j < n$ and also $\ell < \tilde{m} \leq n$.

Recall from (2.17) that $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ is the $(\psi_{\ell, w_0}, \varphi_\pi)$ -**Bessel period** of the Eisenstein series $E(\phi_{\tau \otimes \sigma}, \lambda_s)(g)$, which is given by

$$(3.1) \quad \int_{H_{n-\ell}(F) \backslash H_{n-\ell}(\mathbb{A})} \mathcal{B}^{\psi_{\ell, w_0}}(E(\phi_{\tau \otimes \sigma}, s))(h) \varphi_\pi(h) dh,$$

where the Bessel-Fourier coefficient $\mathcal{B}^{\psi_{\ell, w_0}}(E(\phi_{\tau \otimes \sigma}, s))(h)$ is given, as in (2.11), by

$$(3.2) \quad \mathcal{B}^{\psi_{\ell, w_0}}(E(\phi_{\tau \otimes \sigma}, s))(h) := \int_{N_\ell(F) \backslash N_\ell(\mathbb{A})} E(\phi_{\tau \otimes \sigma}, s)(nh) \psi_{\ell, w_0}^{-1}(n) dn.$$

We first calculate the Bessel-Fourier coefficient $\mathcal{B}^{\psi_{\ell, w_0}}(E(\phi_{\tau \otimes \sigma}, s))$.

3.1. Calculation of Bessel-Fourier coefficients. In order to calculate the Bessel-Fourier coefficient $\mathcal{B}^{\psi_\ell, w_0}(E(\phi_{\tau \otimes \sigma}, s))$, i.e. the integral in (3.2), we assume that the $\text{Re}(s)$ is large, and unfold the Eisenstein series. This leads to consider the double cosets decomposition $P_j \backslash G_n / P_\ell$, whose set of representatives $\epsilon_{\alpha, \beta}$ is explicitly given in [GRS11, Section 4.2]. In our situation, we put it into three cases for discussion.

Case (1): G_n is not the F -split even special orthogonal group. In this case, the set of representatives $\epsilon_{\alpha, \beta}$ of the double coset decomposition $P_j \backslash G_n / P_\ell$ is in bijection with the set of pairs of nonnegative integers

$$(3.3) \quad \{(\alpha, \beta) \mid 0 \leq \alpha \leq \beta \leq j \text{ and } j \leq \ell + \beta - \alpha \leq \tilde{m}\}.$$

Recall that \tilde{m} is the Witt index of (V, q_V) defining G_n .

Case (2-1): G_n is the F -split even special orthogonal group and $\ell + \beta - \alpha < \tilde{m} = n$. In this case, the set of representatives $\epsilon_{\alpha, \beta}$ of the double coset decomposition $P_j \backslash G_n / P_\ell$ is in bijection with the set of pairs of nonnegative integers

$$(3.4) \quad \{(\alpha, \beta) \mid 0 \leq \alpha \leq \beta \leq j \text{ and } j \leq \ell + \beta - \alpha \leq \min\{n-1, \tilde{m}\}\}.$$

Case (2-2): G_n is the F -split even special orthogonal group and $\ell + \beta - \alpha = n$. In this case, there are two double cosets corresponding to each pair (α, β) , and hence we may choose representatives $\epsilon_{\alpha, \beta}$ and $\tilde{\epsilon}_{\alpha, \beta} = w_q \epsilon_{\alpha, \beta} w_q$ of the two double cosets corresponding to such pairs (α, β) .

In all cases, we denote by $P_\ell^{\epsilon_{\alpha, \beta}} := \epsilon_{\alpha, \beta}^{-1} P_j \epsilon_{\alpha, \beta} \cap P_\ell$ the stabilizer in P_ℓ , whose elements have the following form as matrices in $\text{GL}_m(E)$:

$$(3.5) \quad g_\ell^{(\alpha, \beta)} = \begin{pmatrix} a & x_1 & x_2 & y_1 & y_2 & y_3 & z_1 & z_2 & z_3 \\ 0 & b & x_3 & 0 & y_4 & y_5 & 0 & z_4 & z'_2 \\ 0 & 0 & c & 0 & 0 & y_6 & 0 & 0 & z'_1 \\ & & & d & u & v & y'_6 & y'_5 & y'_3 \\ & & & 0 & e & u' & 0 & y'_4 & y'_2 \\ & & & 0 & 0 & d^* & 0 & 0 & y'_1 \\ & & & & & & c^* & x'_3 & x'_2 \\ & & & & & & 0 & b^* & x'_1 \\ & & & & & & 0 & 0 & a^* \end{pmatrix}$$

where the block sizes are determined by $a, a^* \in \text{GL}_\alpha$, $b, b^* \in \text{GL}_{\ell-\alpha-j+\beta}$, $c, c^* \in \text{GL}_{j-\beta}$, $d, d^* \in \text{GL}_{\beta-\alpha}$, and $e \in \text{GL}_{m-2(\ell+\beta-\alpha)}$. In case $i = 0$, GL_i disappears.

The stabilizer in P_j consists of elements of the following form, which are the indicated matrices conjugated by w_q^t :

$$(3.6) \quad g_j^{(\alpha, \beta)} = \epsilon_{\alpha, \beta} g \epsilon_{\alpha, \beta}^{-1} = \begin{pmatrix} a & y_1 & z_1 & x_1 & y_2 & z_2 & x_2 & y_3 & z_3 \\ 0 & d & y'_6 & 0 & u & y'_5 & 0 & v & y'_3 \\ 0 & 0 & c^* & 0 & 0 & x'_3 & 0 & 0 & x'_2 \\ & & & b & y_4 & z_4 & x_3 & y_5 & z'_2 \\ & & & 0 & e & y'_4 & 0 & u' & y'_2 \\ & & & 0 & 0 & b^* & 0 & 0 & x'_1 \\ & & & & & & c & y_6 & z'_1 \\ & & & & & & 0 & d^* & y'_1 \\ & & & & & & 0 & 0 & a^* \end{pmatrix}^{w_q^t}$$

with the block sizes as before and w_q^t being the t -th power of the element w_q for $t = j - \beta$. Also, when (V, q_V) is Hermitian, $w_q = I_m$; when $E = F$ and (V, q_V) is of odd dimension, $w_q = -I_m$; when $E = F$ and anisotropic kernel (V_0, q_{V_0}) is of dimension two, take $w_q = \text{diag}(I_{\tilde{m}}, w_q^0, I_{\tilde{m}})$, where $w_q^0 = \text{diag}\{1, -1\}$; and finally, when $E = F$ and the anisotropic kernel (V_0, q_{V_0}) is a zero space, take $w_q = \text{diag}(I_{\tilde{m}-1}, w_q^0, I_{\tilde{m}-1})$,

where $w_q^0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. Note that $\ell, j < n = [\frac{m}{2}]$, where $m = \dim_E V$ and \tilde{m} is the Witt index of V .

In **Case (2-2)**, i.e. G_n is the F -split even special orthogonal group and $\ell + \beta - \alpha = \tilde{m}$, we have two double cosets corresponding to each pair (α, β) . For the double coset $P_j \epsilon_{\alpha, \beta} P_\ell$, we get exactly the same form for the stabilizer as above. For the other double coset $P_j \tilde{\epsilon}_{\alpha, \beta} P_\ell$, the stabilizer in P_ℓ consists of all elements of the form $(g_\ell^{(\alpha, \beta)})^{w_q}$.

To continue the calculation, we consider further double cosets decomposition $P_\ell^{\epsilon_{\alpha, \beta}} \backslash P_\ell / R_{\ell, w_0}$. Recall that $H_{n-\ell} = \text{U}(q_{W_\ell \cap w_0^\perp})$, $\dim_E V$ and $\dim_E W_\ell \cap w_0^\perp$ have the different parity, and $R_{\ell, w_0} = H_{n-\ell} N_\ell$ with $H_{n-\ell} \cong L_{\ell, w_0}$. By [GRS11, Section 5.1], we may choose a set of representatives of form:

$$(3.7) \quad \eta_{\epsilon, \gamma} := \begin{pmatrix} \epsilon & & \\ & \gamma & \\ & & \epsilon^* \end{pmatrix}$$

where ϵ is a representative in the quotient of Weyl groups

$$W_{\text{GL}_\alpha \times \text{GL}_{\ell-\alpha-t} \times \text{GL}_t} \backslash W_{\text{GL}_\ell}$$

and γ is a representative $P'_w \backslash G_{n-\ell} / H_{n-\ell}$, where P'_w is the maximal parabolic subgroup of $G_{n-\ell}$ defined as follows.

In **Case (1)** or **Case (2-1)**, i.e. when G_n is not the F -split even special orthogonal group or when G_n is the F -split even special orthogonal group with $\ell + \beta - \alpha < n$, then P'_w is the parabolic subgroup of $G_{n-\ell}$, which preserves the standard $\beta - \alpha$ dimensional totally isotropic subspace $V_{\ell, \beta - \alpha}^+$ of W_ℓ , where

$$(3.8) \quad V_{\ell, f}^\pm = \text{Span}_E \{e_{\pm(\ell+1)}, \dots, e_{\pm(\ell+f)}\},$$

for a possible integer f .

In **Case (2-2)**, i.e. when G_n is the F -split even special orthogonal group with $\ell + \beta - \alpha = n$ (with $j, \ell < n$), then, when $w = \epsilon_{\alpha, \beta}$, P'_w is the parabolic subgroup of $G_{n-\ell}$, which preserves $V_{\ell, m-\ell}^+$; and when $w = \tilde{\epsilon}_{\alpha, \beta}$, P'_w is the parabolic subgroup of $G_{n-\ell}$, which preserves $w_q V_{\ell, m-\ell}^+$.

Denote the stabilizer in $H_{n-\ell}$ of the double coset $P'_w \gamma H_{n-\ell}$ with $\eta_{\epsilon, \gamma}$ as defined in (3.7) by

$$(3.9) \quad H_{n-\ell}^{\eta_{\epsilon, \gamma}} = H_{n-\ell}^\gamma = H_{n-\ell} \cap \gamma^{-1} P'_w \gamma = L_{\ell, w_0} \cap \gamma^{-1} P'_w \gamma.$$

With the above preparation, we are ready to calculate the Bessel-Fourier coefficient $\mathcal{B}^{\psi_\ell, w_0}(E(\phi_{\tau \otimes \sigma}, \lambda))(h)$ by assuming that $\text{Re}(s)$ is large so that we are able to unfold the Eisenstein series.

$$\begin{aligned} & \mathcal{B}^{\psi_\ell, w_0}(E(\phi, s))(h) \\ &= \int_{N_\ell(F) \backslash N_\ell(\mathbb{A})} E(\phi, s)(nh) \psi_{\ell, w_0}^{-1}(n) \, dn \\ &= \sum_{\epsilon_{\alpha, \beta} \in \mathcal{E}_{j, \ell}} \int_{N_\ell(F) \backslash N_\ell(\mathbb{A})} \sum_{\delta \in P_\ell^{\epsilon_{\alpha, \beta}}(F) \backslash P_\ell(F)} \lambda \phi(\epsilon_{\alpha, \beta} \delta nh) \psi_{\ell, w_0}^{-1}(n) \, dn, \end{aligned}$$

where $\mathcal{E}_{j, \ell}$ is the set of representatives of $P_j(F) \backslash G_n(F) / P_\ell(F)$. Set $\mathcal{N}_{\alpha, \beta, \ell, w_0}$ to be the set of representatives of $P_\ell^{\epsilon_{\alpha, \beta}}(F) \backslash P_\ell(F) / R_{\ell, w_0}(F)$ and deduce that the above is equal to

$$\sum_{\epsilon_{\alpha, \beta}} \sum_{\eta \in \mathcal{N}_{\alpha, \beta, \ell, w_0}} \int_{N_\ell(F) \backslash N_\ell(\mathbb{A})} \sum_{\delta \in R_{\ell, w_0}^\eta(F) \backslash R_{\ell, w_0}(F)} \lambda \phi(\epsilon_{\alpha, \beta} \eta \delta nh) \psi_{\ell, w_0}^{-1}(n) \, dn.$$

Since $R_{\ell, w_0} = H_{n-\ell} N_\ell$, by re-arranging the summation in δ and the integration of dn , we obtain that the above is equal to

$$\sum_{\epsilon_{\alpha, \beta}} \sum_{\eta} \sum_{\delta \in H_{n-\ell}^\eta(F) \backslash H_{n-\ell}(F)} \int_{N_\ell^\eta(F) \backslash N_\ell(\mathbb{A})} \lambda \phi(\epsilon_{\alpha, \beta} \eta \delta nh) \psi_{\ell, w_0}^{-1}(n) \, dn.$$

By factoring the integration of dn , we obtain that when $\operatorname{Re}(s)$ is large, the Bessel-Fourier coefficient $\mathcal{B}^{\psi_{\ell, w_0}}(E(\phi_{\tau \otimes \sigma}, s))(h)$ is equal to

$$(3.10) \quad \sum_{\epsilon_{\alpha, \beta}; \eta; \delta} \int_{N_{\ell}^{\eta}(\mathbb{A}) \backslash N_{\ell}(\mathbb{A})} \int_{N_{\ell}^{\eta}(F) \backslash N_{\ell}^{\eta}(\mathbb{A})} \lambda \phi(\epsilon_{\alpha, \beta} \eta \delta u n h) \psi_{\ell, w_0}^{-1}(un) du dn.$$

In order to determine the summands in (3.10), we need the following two lemmas, which are the global version of Propositions 5.1 and 5.2 in [GRS11, Chapter 5].

Lemma 3.1. *If $\alpha > 0$, then the inner integral in (3.10) has the following property:*

$$\int_{N_{\ell}^{\eta}(F) \backslash N_{\ell}^{\eta}(\mathbb{A})} \lambda \phi(\epsilon_{\alpha, \beta} \eta u n h) \psi_{\ell, w_0}^{-1}(un) du = 0$$

for all choices of data.

Proof. If there exists a simple root subgroup U of Z_{ℓ} such that $\epsilon U \epsilon^{-1}$ lies inside $U_{\alpha, \ell - \alpha - t, t}$ for some $\epsilon \in W_{\mathrm{GL}_{\alpha} \times \mathrm{GL}_{\ell - \alpha - t} \times \mathrm{GL}_t} \backslash W_{\mathrm{GL}_{\ell}}$, then the subgroup $\epsilon_{\alpha, \beta} \eta_{\epsilon, \gamma} U(\epsilon_{\alpha, \beta} \eta_{\epsilon, \gamma})^{-1}$ lies inside U_j . Since the automorphic function $\lambda \phi$ is invariant on $U_j(\mathbb{A})$ and ψ_{ℓ, w_0} is not trivial on $U(\mathbb{A})$,

$$\begin{aligned} & \int_{U(F) \backslash U(\mathbb{A})} \lambda \phi(\epsilon_{\alpha, \beta} \eta z u n h) \psi_{\ell, w_0}^{-1}(z) dz \\ &= \lambda \phi(\epsilon_{\alpha, \beta} \eta u n h) \cdot \int_{E \backslash \mathbb{A}_E} \psi^{-1}(x) dx \end{aligned}$$

is identically zero.

If for each simple root subgroup U of GL_{ℓ} , $\epsilon U \epsilon^{-1}$ does not lie inside $U_{\alpha, \ell - \alpha - t, t}$, then according to [GRS11, Lemma 5.1], we choose, under the action of the Weyl group of $M_{\alpha, \ell - \alpha - t, t}$,

$$(3.11) \quad \epsilon = \begin{pmatrix} & I_{\alpha} \\ I_{\ell - \alpha - t} & \\ I_t & \end{pmatrix}.$$

Since $\alpha \neq 0$ ($\ell < \tilde{m}$), we choose a nontrivial subgroup S of N_{ℓ} consisting of elements of form

$$\begin{pmatrix} I_{\ell - \alpha} & & & \\ & I_{\alpha} & y & * \\ & & I_{m - 2\ell} & y' \\ & & & I_{\alpha} \\ & & & & I_{\ell - \alpha} \end{pmatrix}$$

where $(\tilde{y}_1 \ \tilde{y}_2 \ \tilde{y}_3) = (0_{r \times (\beta - \alpha)} \ y_2 \ y_3)(w_q^{t'} \gamma)^{-1}$, and y_2 and y_3 are of size $\alpha \times (m - 2(\ell + \beta - \alpha))$ and $\alpha \times (\beta - \alpha)$, respectively; and when G_n is

split, even orthogonal, $\ell + \beta - \alpha = n$ and the representative $w = \epsilon_{\alpha, \beta}^{w_q}$, we have that $t' = 1$, otherwise, we always have that $t' = 0$. Since w_0 is anisotropic, w_0 is not orthogonal to $V_0 \oplus V_{\ell, \beta - \alpha}^-$ and ψ_{ℓ, w_0} is not trivial on $S(\mathbb{A})$. By (3.6), we have $(\epsilon_{\alpha, \beta} \eta_{\epsilon, \gamma}) S(\epsilon_{\alpha, \beta} \eta_{\epsilon, \gamma})^{-1}$ lies inside U_j and then

$$\begin{aligned} & \int_{S(F) \backslash S(\mathbb{A})} \lambda \phi(\epsilon_{\alpha, \beta} \eta x u n h) \psi_{\ell, w_0}^{-1}(x) dx \\ &= \lambda \phi(\epsilon_{\alpha, \beta} \eta u n h) \cdot \int_{S(F) \backslash S(\mathbb{A})} \psi_{\ell, w_0}^{-1}(x) dx \end{aligned}$$

is identically zero. This proves the lemma. \square

By Lemma 3.1 and (3.10), when $\text{Re}(s)$ is large, the Bessel-Fourier coefficient $\mathcal{B}^{\psi_{\ell, w_0}}(E(\phi_{\tau \otimes \sigma}, \lambda))(h)$ is equal to

$$(3.12) \quad \sum_{\epsilon_{0, \beta}; \delta} \int_{N_{\ell}^{\eta}(\mathbb{A}) \backslash N_{\ell}(\mathbb{A})} \int_{N_{\ell}^{\eta}(F) \backslash N_{\ell}^{\eta}(\mathbb{A})} \lambda \phi(\epsilon_{0, \beta} \eta \delta u n h) \psi_{\ell, w_0}^{-1}(un) du dn.$$

In particular, we may choose the ϵ in (3.11), which is part of the representation $\eta_{\epsilon, \gamma}$ in (3.7), to be of the form: $\epsilon = \begin{pmatrix} I_{\ell-t} \\ I_t \end{pmatrix}$. Note that ϵ is one of the representatives of $W_{\text{GL}_{\alpha} \times \text{GL}_{\ell-\alpha-t} \times \text{GL}_t} \backslash W_{\text{GL}_{\ell}}$. The following lemma will help us to eliminate more terms in (3.12).

Lemma 3.2. *If $\beta > \max\{j - \ell, 0\}$ and γw_0 is not orthogonal to $V_{\ell, \beta}^-$ for $\gamma \in P'_w \backslash G_{n-\ell}/H_{n-\ell}$, then the inner integral in (3.12) has the property:*

$$\int_{N_{\ell}^{\eta}(F) \backslash N_{\ell}^{\eta}(\mathbb{A})} \lambda \phi(\epsilon_{0, \beta} \eta_{\epsilon, \gamma} u n h) \psi_{\ell, w_0}^{-1}(un) du = 0$$

for all choices of data.

Proof. Consider the subgroup S of N_{ℓ} consisting of elements of form

$$\begin{pmatrix} I_t & & & & \\ & I_{\ell-t} & y & * & \\ & & I_{m-2\ell} & y' & \\ & & & I_{\ell-t} & \\ & & & & I_{\beta} \end{pmatrix},$$

where $y = (0_{(\ell-t) \times (m-2\ell-\beta)} \ y_5)(w_q^{t'} \gamma)^{-1}$ with t' as defined before. and y_5 is of size $(\ell - t) \times \beta$. By $\ell - t = \ell - j + \beta > 0$ and $\beta > 0$, y_5 is not trivial. Since γw_0 is not orthogonal to $V_{\ell, \beta}^-$, ψ_{ℓ, w_0} is not trivial on

$S(\mathbb{A}_F)$. By (3.6), ϕ is invariant on $(\epsilon_{0,\beta}\eta_{\epsilon,\gamma})S(\mathbb{A})(\epsilon_{0,\beta}\eta_{\epsilon,\gamma})^{-1}$. it follows that as an inner integration, the following integral

$$\begin{aligned} & \int_{S(F)\backslash S(\mathbb{A})} \lambda\phi(\epsilon_{0,\beta}\eta_{\epsilon,\gamma}xunh)\psi_{\ell,w_0}^{-1}(x) dx \\ &= \lambda\phi(\epsilon_{0,\beta}\eta_{\epsilon,\gamma}unh) \cdot \int_{S(F)\backslash S(\mathbb{A})} \psi_{\ell,w_0}^{-1}(x) dx \end{aligned}$$

is identically zero. This finishes the proof. \square

We summarize the above calculation as

Proposition 3.3. *For $\text{Re}(s)$ large, the Bessel-Fourier coefficient of the Eisenstein series, $\mathcal{B}^{\psi_{\ell,w_0}}(E(\phi_{\tau\otimes\sigma}, \lambda))(h)$, is equal to*

$$\sum_{\epsilon_\beta} \sum_{\eta} \sum_{\delta} \int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})} \int_{N_\ell^\eta(F)\backslash N_\ell^\eta(\mathbb{A})} \lambda\phi(\epsilon_\beta\eta\delta unh)\psi_{\ell,w_0}^{-1}(un) du dn,$$

where

- $\epsilon_\beta = \epsilon_{0,\beta} \in \mathcal{E}_{j,\ell}$, the set of representatives of all double cosets in $P_j(F)\backslash G_n(F)/P_\ell(F)$, with $\alpha = 0$ and the properties that if $\beta > \max\{j - \ell, 0\}$, then γw_0 is orthogonal to $V_{\ell,\beta}^-$ for $\gamma \in P'_w(F)\backslash G_{n-\ell}(F)/H_{n-\ell}(F)$; or otherwise, $\beta = \max\{j - \ell, 0\}$;
- $\eta = \text{diag}(\epsilon, \gamma, \epsilon^*)$ belongs to $\mathcal{N}_{\beta,\ell,w_0}$ with $\alpha = 0$, the set of representatives of $P_\ell^{\epsilon_\beta}(F)\backslash P_\ell(F)/R_{\ell,w_0}(F)$ with $\epsilon = \begin{pmatrix} & I_{\ell-t} \\ I_t & \end{pmatrix}$ and $t = j - \beta$;
- δ belongs to $H_{n-\ell}^\eta(F)\backslash H_{n-\ell}(F)$.

We are going to apply the formula in Proposition 3.3 to the calculation of the global zeta integral $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$ and use the cuspidality of φ_π to prove that the global zeta integral $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$ is eulerian.

3.2. Global zeta integrals. By applying Proposition 3.3 to the global zeta integral in (3.1), we get

(3.13)

$$\begin{aligned} & \mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0}) \\ &= \int_{H_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})} \mathcal{B}^{\psi_{\ell,w_0}}(E(\phi_{\tau\otimes\sigma}, s))(h)\varphi(h) dh \\ &= \sum_{\epsilon_\beta; \eta; \delta} \int_{[H_{n-\ell}]} \varphi(h) \int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})} \int_{[N_\ell^\eta]} \lambda\phi(\epsilon_\beta\eta\delta unh)\psi_{\ell,w_0}^{-1}(un) du dn dh \end{aligned}$$

where $[H_{n-\ell}] := H_{n-\ell}(F) \backslash H_{n-\ell}(\mathbb{A})$ and $[N_\ell^\eta] := N_\ell^\eta(F) \backslash N_\ell^\eta(\mathbb{A})$; and the summations $\sum_{\epsilon_\beta; \eta; \delta}$ and other conditions for the representatives are given in Proposition 3.3.

We combine the summation on δ and the integration dh and obtain that $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ is equal to
(3.14)

$$\sum_{\epsilon_\beta; \eta} \int_{H_{n-\ell}^\eta(F) \backslash H_{n-\ell}(\mathbb{A})} \varphi(h) \int_n \int_{[N_\ell^\eta]} \lambda \phi(\epsilon_\beta \eta u n h) \psi_{\ell, \kappa}^{-1}(un) du dn dh,$$

where the integration \int_n is over $N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})$. The following lemma is to make use of the cuspidality of φ_π .

Lemma 3.4. *Let $\alpha = 0$ and γ be a representative in $P'_w \backslash G_{n-\ell} / H_{n-\ell}$. For a representative $\eta = \eta_{\epsilon, \gamma}$, if the stabilizer $H_{n-\ell}^\eta$ is a proper maximal parabolic subgroup of $H_{n-\ell}$, then the corresponding summand in (3.14) has the property:*

$$\int_{H_{n-\ell}^\eta(F) \backslash H_{n-\ell}(\mathbb{A})} \varphi(h) \int_n \int_{[N_\ell^\eta]} \lambda \phi(\epsilon_\beta \eta_{\epsilon, \gamma} u n h) \psi_{\ell, w_0}^{-1}(un) du dn dh = 0$$

for all choices of data.

Proof. Let $H_{n-\ell}^\eta = M'U'$, where U' is the unipotent radical of the parabolic subgroup $H_{n-\ell}^\eta$ of $H_{n-\ell}$. Since ϕ is left-invariant with respect to the image under the adjoint action by $\epsilon_{0, \beta} \eta_{\epsilon, \gamma}$ of the unipotent radical $U'(\mathbb{A})$ of $H_{n-\ell}^\eta(\mathbb{A})$, we deduce that

$$\begin{aligned} & \int_{H_{n-\ell}^\eta(F) \backslash H_{n-\ell}(\mathbb{A})} \varphi(h) \int_n \int_{[N_\ell^\eta]} \lambda \phi(\epsilon_\beta \eta_{\epsilon, \gamma} u n h) \psi_{\ell, w_0}^{-1}(un) du dn dh \\ &= \int_h \int_{[U']} \varphi(u'h) du' \int_n \int_{[N_\ell^\eta]} \lambda \phi(\epsilon_\beta \eta_{\epsilon, \gamma} u n h) \psi_{\ell, w_0}^{-1}(un) du dn dh \end{aligned}$$

where \int_h is over $M'(F)U'(\mathbb{A}) \backslash H_{n-\ell}(\mathbb{A})$. By the cuspidality of π , we have that

$$\int_{U'(F) \backslash U'(\mathbb{A})} \varphi(u'h) du' = 0,$$

and hence the whole integral is zero. This proves the lemma. \square

By Proposition 3.3, the representatives ϵ_β have the restrictions that either $\beta = \max\{0, j-\ell\}$ or $\beta > \max\{0, j-\ell\}$ with γw_0 being orthogonal to $V_{\ell, \beta}^-$ for $\gamma \in P'_w(F) \backslash G_{n-\ell}(F) / H_{n-\ell}(F)$. Next, we discuss the double cosets decomposition $\gamma \in P'_w(F) \backslash G_{n-\ell}(F) / H_{n-\ell}(F)$.

Lemma 3.5 (Proposition 4.4, [GRS11]). *Let X be a non-trivial totally isotropic subspace of W_ℓ and P be the maximal parabolic subgroup of $G_{n-\ell}$ preserving X . Then*

- (1) *If $\dim_E X < \text{Witt}(W_\ell)$, then the set $P \backslash G_{n-\ell} / H_{n-\ell}$ consists of two elements.*
- (2) *Assume that $\text{Witt}(w_0^\perp) = \dim_E X = \text{Witt}(W_\ell)$.*
 - (a) *If $G_{n-\ell}$ is unitary, then $P \backslash G_{n-\ell} / H_{n-\ell}$ consists of two elements.*
 - (b) *If $G_{n-\ell}$ is orthogonal and $\dim W_\ell \geq 2 \dim X + 2$, then $P \backslash G_{n-\ell} / H_{n-\ell}$ consists of two elements.*
 - (c) *If $G_{n-\ell}$ is orthogonal and $\dim W_\ell = 2 \dim X + 1$, then $P \backslash G_{n-\ell} / H_{n-\ell}$ consists of three elements.*
- (3) *If $\dim_E X = \text{Witt}(W_\ell)$ and $\text{Witt}(w_0^\perp) = \dim_E X - 1$, then $P \backslash G_{n-\ell} / H_{n-\ell}$ consists of one element.*
- (4) *If $\dim_E W_\ell = 2 \dim_E X$, then $\text{Witt}(w_0^\perp) = \dim X - 1$, and, in particular, $P \backslash G_{n-\ell} / H_{n-\ell}$ consists of one element.*

We consider the case when $G_{n-\ell}$ is not the F -split even orthogonal group or the case when $G_{n-\ell}$ is the F -split even orthogonal group with $\ell + \beta < n$. In these cases, we must have that $\dim X = \beta$.

If $\ell + \beta < \tilde{m}$, then $P'_w \backslash G_{n-\ell} / H_{n-\ell}$ consists of two elements. It remains to consider that $\ell + \beta = \tilde{m}$. If $\ell + \beta < n$, we must have that $\ell + \beta = \tilde{m} < n$ and hence $G_{n-\ell}$ can not be the F -split even special orthogonal group.

In this case $\ell + \beta = \tilde{m} < n$, if $G_{n-\ell}$ is an F -quasisplit even unitary group, then $\text{Witt}(W_\ell \cap y_\kappa^\perp) = \text{Witt}(W_\ell) - 1$ and $P'_w \backslash G_{n-\ell} / H_{n-\ell}$ has only one element; if $G_{n-\ell}$ is an odd special orthogonal group, then

$$\# P'_w \backslash G_{n-\ell} / H_{n-\ell} = \begin{cases} 3, & \text{if } \text{Witt}(w_0^\perp \cap W_\ell) = \tilde{m} - \ell, \\ 1, & \text{if } \text{Witt}(w_0^\perp \cap W_\ell) = \tilde{m} - \ell - 1; \end{cases}$$

and if $G_{n-\ell}$ is an F -quasisplit even special orthogonal group (with $\dim V_0 = 2$) or an F -quasisplit odd unitary group, then

$$\# P'_w \backslash G_{n-\ell} / H_{n-\ell} = \begin{cases} 2, & \text{if } \text{Witt}(w_0^\perp \cap W_\ell) = \tilde{m} - \ell, \\ 1, & \text{if } \text{Witt}(w_0^\perp \cap W_\ell) = \tilde{m} - \ell - 1. \end{cases}$$

It remains to consider the case when $G_{n-\ell}$ is an F -split even special orthogonal group with $\ell + \beta = n$. In this case, $P'_w \backslash G_{n-\ell} / H_{n-\ell}$ consists of two elements.

Then, we apply Lemmas 3.4 and 3.5 to find the nonvanishing summand in the summation (3.14).

For $\max\{0, j - \ell\} \leq \beta < \tilde{m} - \ell$, $P'_w \backslash G_{n-\ell} / H_{n-\ell}$ consists of two elements γ_1 and γ_2 such that $\gamma_1 w_0$ is orthogonal to $V_{\ell, \beta}^-$ and $\gamma_2 w_0$ is not orthogonal to $V_{\ell, \beta}^-$. If γw_0 is orthogonal to $V_{\ell, \beta}^-$, the stabilizer $H_{n-\ell}^\gamma = H_{n-\ell}^\eta$ is a maximal parabolic subgroup of $H_{n-\ell}$, which preserves the isotropic subspace $w_q^t V_{\ell, \beta}^+ \cap w_0^\perp$.

In this case, by Lemmas 3.2 and 3.4, there may be left with nonzero summands in the summation (3.14), which are with the representative ϵ_β for $\beta = \max\{0, j - \ell\}$ and with the representative $\eta = \eta_{\epsilon, \gamma}$ having the property that γw_0 is not orthogonal to $V_{\ell, \beta}^-$.

For $\beta = \tilde{m} - \ell$, there are six different cases. Also, we have that $\beta = \tilde{m} - \ell > \max\{0, j - \ell\}$.

If G_n is the F -split even special orthogonal group, then there are two (P_j, P_ℓ) -double cosets corresponding to the pair $(0, \beta)$ and the chosen representatives are $\epsilon_{0, \beta}$ and $\tilde{\epsilon}_{0, \beta}$. For these two cases, their stabilizer preserves two maximal isotropic subspace of W_ℓ with different orientations, and $P'_w \backslash G_{n-\ell} / H_{n-\ell}$ consists of one element in both cases with its stabilizer $H_{n-\ell}^\gamma = H_{n-\ell}^\eta$ being a maximal parabolic subgroup. Hence by Lemma 3.4, the corresponding summands are all zero.

If G_n is not the F -split even special orthogonal group and $\text{Witt}(W_\ell \cap y_\kappa^\perp) = \text{Witt}(W_\ell) - 1$, there is only one double coset whose stabilizer is a maximal parabolic subgroup of $H_{n-\ell}$. Hence by Lemma 3.4, the corresponding summand is zero.

If $\text{Witt}(W_\ell \cap y_\kappa^\perp) = \text{Witt}(W_\ell)$ and G_n is the odd unitary group or F -quasi-split even special orthogonal group, the stabilizers are similar to the case $\beta < \tilde{m} - \ell$ as discussed above. Hence by Lemmas 3.2 and 3.4, the corresponding summands are all zero.

If G_n is the odd special orthogonal group and $\text{Witt}(W_\ell \cap y_\kappa^\perp) = \text{Witt}(W_\ell) - 1$, then $P'_w \backslash G_{n-\ell} / H_{n-\ell}$ consists of three elements and the representatives are chosen in [GRS11, (4.33)]. Two stabilizers are maximal parabolic subgroups of $H_{n-\ell}$, and the third representative γ satisfies the property that γw_0 is not orthogonal to $V_{\ell, \beta}^-$. Hence by Lemmas 3.2 and 3.4, the corresponding summands are all zero.

By the discussions above, we deduce that the corresponding summands are all zero, because of Lemmas 3.2 and 3.4.

In conclusion, we are left with the case where $\beta = \max\{0, j - \ell\}$ and γ with the property that the corresponding stabilizer is not a proper maximal parabolic subgroup of $H_{n-\ell}^\gamma$, i.e. γw_0 is not orthogonal to $V_{\ell, \beta}^-$.

In this case, the representative $\eta = \eta_{\epsilon, \gamma}$ is uniquely determined by $\beta = \max\{0, j - \ell\}$. In fact, if $j \leq \ell$, then $\beta = 0$. It follows that $\eta = \eta_{\epsilon, \gamma}$

with $\gamma = I_{m-2\ell}$ and

$$(3.15) \quad \epsilon = \begin{pmatrix} & I_{\ell-j} \\ I_j & \end{pmatrix};$$

and if $j > \ell$, then $\beta = j - \ell$. It implies that $\eta = \eta_{\epsilon, \gamma}$ with $\epsilon = I_\ell$ and

$$(3.16) \quad \gamma = \begin{pmatrix} & I_{j-\ell} & & \\ I_{\tilde{m}-j} & & & \\ & I_{V_0} & & \\ & & I_{\tilde{m}-j} & \\ & & & I_{j-\ell} \end{pmatrix}.$$

Therefore, we are left with only one summand in the summation (3.14) with the above representative, accordingly.

Next we are going to write the only integral more explicitly and get ready to prove that it is eulerian in the next subsection.

If $j \leq \ell$, then $\beta = 0$. In this case we have that $P'_w = G_{n-\ell}$ and $H_{n-\ell}^\gamma = H_{n-\ell}$ with ϵ and γ given above. Then the global zeta integral in (3.14) has the following expression:

$$(3.17) \quad \mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0}) = \int_{[H_{n-\ell}]} \varphi(h) \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \lambda \phi(\epsilon_{0,0} \eta u n h) \psi_{\ell, w_0}^{-1}(un) du dn dh.$$

where $[H_{n-\ell}] := H_{n-\ell}(F) \backslash H_{n-\ell}(\mathbb{A})$ and $[N_\ell^\eta] := N_\ell^\eta(F) \backslash N_\ell^\eta(\mathbb{A})$. The stabilizers are, respectively, given by

$$(3.18) \quad R_{\ell, w_0}^\eta = \begin{pmatrix} c & 0 & 0 & 0 & 0 \\ & b & y_4 & z_4 & 0 \\ & & e & y'_4 & 0 \\ & & & b^* & 0 \\ & & & & c^* \end{pmatrix}$$

with c, c^* being of size $j \times j$, b, b^* of size $(\ell - j) \times (\ell - j)$, and e of size $(m - 2\ell) \times (m - 2\ell)$; and

$$(3.19) \quad (\epsilon_{0,0} \eta_{\epsilon, \gamma}) R_{\ell, w_0}^\eta (\epsilon_{0,0} \eta_{\epsilon, \gamma})^{-1} = \begin{pmatrix} c^* & 0 & 0 & 0 & 0 \\ & b & y_4 & z_4 & 0 \\ & & e & y'_4 & 0 \\ & & & b^* & 0 \\ & & & & c \end{pmatrix}$$

with $c \in Z_j$ and $b \in Z_{\ell-j}$. (Z_f is the maximal upper-triangular unipotent subgroup of GL_f .)

If $j > \ell$, then $\beta = j - \ell$. In this case, $\epsilon = I_\ell$ and γ is given in (3.16). The double coset decomposition $P'_w \backslash G_{n-\ell} / H_{n-\ell}$ produces two representatives which, as given in [GRS11, Section 4.4], are $\gamma = I_{m-2\ell}$ and the γ as given in (3.16).

For the representative $\gamma = I_{m-2\ell}$, the corresponding stabilizer $H_{n-\ell}^\gamma$ is a proper maximal parabolic subgroup. Then, the corresponding integral in (3.14) is zero by Lemma 3.4.

Now for the γ as given in (3.16), we have that the global zeta integral is expressed as

$$(3.20) \quad \mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0}) = \int_{H_{n-\ell}^\eta(F) \backslash H_{n-\ell}(\mathbb{A})} \varphi(h) \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \int_{[N_\ell^\eta]} \lambda \phi(\epsilon_\beta \eta u n h) \psi_{\ell, w_0}^{-1}(un) du dn dh,$$

where $[N_\ell^\eta] = N_\ell^\eta(F) \backslash N_\ell^\eta(\mathbb{A})$. The stabilizers are given, respectively,

$$(3.21) \quad \eta_{\epsilon, \gamma}^{-1} R_{\ell, w_0}^\eta \eta_{\epsilon, \gamma} = \begin{pmatrix} c & 0 & 0 & y_6 & 0 \\ & d & u & v & y'_6 \\ & & e & u' & 0 \\ & & & d^* & 0 \\ & & & & c^* \end{pmatrix}$$

with c, c^* being of size $\ell \times \ell$, d, d^* of size $(j - \ell) \times (j - \ell)$, and e of size $(m - 2j) \times (m - 2j)$; and

$$(3.22) \quad (\epsilon_{0, \beta} \eta_{\epsilon, \gamma}) R_{\ell, w_0}^\eta (\epsilon_{0, \beta} \eta_{\epsilon, \gamma})^{-1} = \begin{pmatrix} d & y'_6 & u & 0 & v \\ & c^* & 0 & 0 & 0 \\ & & e & 0 & u' \\ & & & c & y_6 \\ & & & & d^* \end{pmatrix}$$

where $c \in Z_\ell$.

We conclude this subsection with the following proposition which summarizes the calculations discussed up to this point.

Proposition 3.6. *Take notation as given above. If $j \leq \ell$, then $\beta = 0$ and the global zeta integral has the following expression:*

$$\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0}) = \int_{[H_{n-\ell}]} \varphi(h) \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \int_{[N_\ell^\eta]} \lambda \phi(\epsilon_{0,0} \eta u n h) \psi_{\ell, w_0}^{-1}(un) du dn dh,$$

where $[H_{n-\ell}] := H_{n-\ell}(F) \backslash H_{n-\ell}(\mathbb{A})$ and $[N_\ell^\eta] := N_\ell^\eta(F) \backslash N_\ell^\eta(\mathbb{A})$; and with $\eta = \eta_{\epsilon, \gamma}$ given explicitly above. If $j > \ell$, then $\beta = j - \ell$ and the

global zeta integral has the following expression:

$$\begin{aligned} \mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0}) &= \int_{H_{n-\ell}^\eta(F) \backslash H_{n-\ell}(\mathbb{A})} \varphi(h) \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \\ &\quad \int_{[N_\ell^\eta]} \lambda \phi(\epsilon_{0,\beta} \eta u n h) \psi_{\ell, w_0}^{-1}(un) du dn dh, \end{aligned}$$

where $[N_\ell^\eta] = N_\ell^\eta(F) \backslash N_\ell^\eta(\mathbb{A})$; and with $\eta = \eta_{\epsilon, \gamma}$ given explicitly above.

We are going to show that the global zeta integrals are eulerian based on Proposition 3.6. This is done for the two cases, separately.

3.3. Eulerian products: $0 < \ell < j$ case. In this case, we have that $\beta = j - \ell$. By Proposition 3.6, the global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ is equal to the following integral

$$(3.23) \quad \int_{H_{n-\ell}^\eta(F) \backslash H_{n-\ell}(\mathbb{A})} \varphi(h) \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \int_{[N_\ell^\eta]} \lambda \phi(\epsilon_{0,\beta} \eta u n h) \psi_{\ell, w_0}^{-1}(un) du dn dh,$$

where $[N_\ell^\eta] = N_\ell^\eta(F) \backslash N_\ell^\eta(\mathbb{A})$; and with $\eta = \eta_{\epsilon, \gamma}$ given explicitly above.

First, we want to understand the Fourier coefficient of $\lambda \phi$:

$$(3.24) \quad \int_{[N_\ell^\eta]} \lambda \phi(\epsilon_{0,\beta} \eta u h) \psi_{\ell, w_0}^{-1}(u) du.$$

By conjugating the element $\epsilon_{0,\beta} \eta$ across the variable u and changing the variable by

$$(\epsilon_{0,\beta} \eta) u (\epsilon_{0,\beta} \eta)^{-1} \mapsto \hat{z}',$$

the Fourier coefficient in (3.24) reduces to

$$(3.25) \quad \int_{[Z'_\ell]} \lambda \phi(\hat{z}' \epsilon_{0,\beta} \eta h) \psi_{\ell, w_0}^{-1}((\epsilon_{0,\beta} \eta)^{-1} \hat{z}' (\epsilon_{0,\beta} \eta)) dz',$$

where $Z'_\ell = (\epsilon_{0,\beta} \eta) N_\ell^\eta (\epsilon_{0,\beta} \eta)^{-1}$, whose elements z' are of form

$$(3.26) \quad z' = \begin{pmatrix} I_\beta & y \\ & z \end{pmatrix} \in \text{Res}_{E/F}(\text{GL}_j)$$

with $z \in Z_\ell$, where $g \in \text{Res}_{E/F}(\text{GL}_j)$ is identified with its embedding $\hat{g} = (g, I_{m-2j})$ into the Levi subgroup $\text{Res}_{E/F}(\text{GL}_j) \times G_{n-j}$ of G_n . It follows from the choice of the representatives $\epsilon_{0,\beta}$ and η that the character has following expression:

$$(3.27) \quad \psi_{\ell, w_0}^{-1}((\epsilon_{0,\beta} \eta)^{-1} \hat{z}' (\epsilon_{0,\beta} \eta)) = \psi(z_{1,2} + \cdots + z_{\ell-1,\ell} + (-1)^{m+1} \frac{\kappa}{2} y_{\beta,1}),$$

where $z = (z_{e,f})_{\ell \times \ell}$. If we write elements z' of Z'_ℓ as $z' = (z'_{e,f})_{j \times j}$, then this character can be written as

$$(3.28) \quad \psi_{Z'_\ell, \kappa}(z') := \psi((-1)^{m+1} \frac{\kappa}{2} z_{\beta, \beta+1} + z_{\beta+1, \beta+2} + \cdots + z_{j-1, j}).$$

In this way, the Fourier coefficient in (3.25) can be written as

$$(3.29) \quad \phi_\lambda^{\psi_{Z'_\ell, \kappa}}(h) := \int_{[Z'_\ell]} \lambda \phi(\hat{z}' h) \psi_{Z'_\ell, \kappa}(z') dz'.$$

Hence the global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$, which is expressed as in (3.23), is equal to the following integral

$$(3.30) \quad \int_{H_{n-\ell}^\eta(F) \backslash H_{n-\ell}(\mathbb{A})} \varphi(h) \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \phi_\lambda^{\psi_{Z'_\ell, \kappa}}(\epsilon_{0, \beta} \eta n h) \psi_{\ell, w_0}^{-1}(n) dn dh,$$

with $\eta = \eta_{\epsilon, \gamma}$ given explicitly above.

Next, we want to understand the structure of the subgroup $H_{n-\ell}^\eta$. By (3.9), $H_{n-\ell}^\eta = H_{n-\ell} \cap \gamma^{-1} P'_w \gamma$ with $\eta = \eta_{\epsilon, \gamma}$, and $P'_w = G_{n-\ell} \cap P_\ell^{\epsilon_{0, \beta}}$ is the parabolic subgroup of $G_{n-\ell}$, preserving the totally isotropic subspace $V_{\ell, \beta}^+$ as in (3.8). Denote by

$$P_w'^\eta = P'_w \cap \eta H_{n-\ell} \eta^{-1} = P'_w \cap \gamma H_{n-\ell} \gamma^{-1}.$$

Then the elements of $P_w'^\eta$ are of form:

$$(3.31) \quad \begin{pmatrix} I_\ell & & & & & \\ & d & d_1 & u & v_1 & v \\ & & 1 & 0 & 0 & v'_1 \\ & & & e & 0 & u' \\ & & & & 1 & d'_1 \\ & & & & & d^* \\ & & & & & & I_\ell \end{pmatrix}$$

with $d_1 + (-1)^{m+1} \frac{\kappa}{2} v_1 = 0$, where d_1 and v_1 are column vectors of size $\beta - 1$; d, d^* are of size $(\beta - 1) \times (\beta - 1)$; and e belongs to G_{n-j} . Hence we have

$$(3.32) \quad P_w'^\eta = (\mathrm{GL}(V_{\ell, \beta-1}^+) \times G_{n-j}) \rtimes U^\eta(V_{\ell, \beta-1}^+),$$

where $U^\eta(V_{\ell, \beta-1}^+)$ is the subgroup of $U(V_{\ell, \beta-1}^+)$ consisting elements which fixes the vector γy_κ . Here $U(V_{\ell, \beta-1}^+)$ is the unipotent radical of the parabolic subgroup $P(V_{\ell, \beta-1}^+)$ of $G_{n-\ell}$ preserving the totally isotropic subspace $V_{\ell, \beta-1}^+$.

Let $Q_{\beta-1, \eta}$ be the parabolic subgroup of $H_{n-\ell}$, which preserves the totally isotropic subspace $(\eta^{-1} V_{\ell, \beta}^+) \cap y_\kappa^\perp$ of $W_\ell \cap y_\kappa^\perp$ and has the Levi

decomposition

$$Q_{\beta-1,\eta} = L_{\beta-1,\eta} V_{\beta-1,\eta}.$$

Recall that the space $W_\ell \cap y_\kappa^\perp$ has the polar decomposition

$$W_\ell \cap y_\kappa^\perp = V_{\ell,\tilde{m}-\ell-1}^+ \oplus W_0 \oplus V_{\ell,\tilde{m}-\ell-1}^-,$$

where W_0 is a non-degenerate subspace of $W_\ell \cap y_\kappa^\perp$ with the same anisotropic kernel as $W_\ell \cap y_\kappa^\perp$ and with $\dim_E W_0 = \dim_E V_0 + 1 \leq 3$. In particular, if $w_0 = y_\kappa = e_{\tilde{m}} + (-1)^{m+1} \frac{\kappa}{2} e_{-\tilde{m}}$, then $W_0 = \text{Span}\{y_{-\kappa}\} \oplus V_0$, otherwise, W_0 has Witt index 1. Then it is easy to check that

$$(\eta^{-1} V_{\ell,\beta}^+) \cap y_\kappa^\perp = \text{Span}\{e_{\tilde{m}-j+\ell+1}, \dots, e_{\tilde{m}-1}\} = V_{\tilde{m}-\beta,\beta-1}^+,$$

and

$$L_{\beta-1,\eta} = \text{GL}(V_{\tilde{m}-\beta,\beta-1}^+) \times H_{n-j+1},$$

where $H_{n-j+1} := \text{U}(q_{W_{j-1} \cap y_\kappa^\perp})$ with

$$W_{j-1} \cap y_\kappa^\perp = V_{\ell,\tilde{m}-j}^+ \oplus W_0 \oplus V_{\ell,\tilde{m}-j}^-.$$

It follows that

$$\text{GL}(V_{\tilde{m}-\beta,\beta-1}^+) = \text{GL}((\eta^{-1} V_{\ell,\beta}^+) \cap y_\kappa^\perp) = \eta^{-1} \text{GL}(V_{\ell,\beta-1}^+) \eta \subset H_{n-\ell}^\eta,$$

and

$$V_{\beta-1,\eta} = \eta^{-1} U^\eta(V_{\ell,\beta-1}^+) \eta \subset H_{n-\ell}^\eta.$$

It is easy to check that

$$\eta^{-1} W_j = V_{\ell,\tilde{m}-j}^+ \oplus V_0 \oplus V_{\ell,\tilde{m}-j}^- = y_{-\kappa}^\perp \cap (W_{j-1} \cap y_\kappa^\perp).$$

Hence we have

$$\text{U}(q_{\eta^{-1}W_j}) = \eta^{-1} \text{U}(q_{W_j}) \eta = \eta^{-1} G_{n-j} \eta \subset H_{n-\ell}^\eta.$$

Putting together all these subgroups, we obtain the structure of $H_{n-\ell}^\eta$:

$$(3.33) \quad H_{n-\ell}^\eta = (\text{GL}(V_{\tilde{m}-\beta,\beta-1}^+) \times \text{U}(q_{\eta^{-1}W_j})) \rtimes V_{\beta-1,\eta}.$$

Finally, we are ready to consider partial Fourier expansion of the cuspidal automorphic forms φ_π on $H_{n-\ell}(\mathbb{A})$. Let $Z_{\ell,\beta-1}^\eta$ be the maximal unipotent subgroup of $\text{GL}(V_{\tilde{m}-\beta,\beta-1}^+)$ consisting of elements of following type:

$$\eta^{-1} \begin{pmatrix} I_\ell & & & & \\ & d & & & \\ & & I_{m-2j+2} & & \\ & & & d^* & \\ & & & & I_\ell \end{pmatrix} \eta$$

with $d \in Z_{\beta-1}$. Then $N_{\ell,\beta-1}^\eta = Z_{\ell,\beta-1}^\eta V_{\beta-1,\eta}$ is a unipotent subgroup of $H_{n-\ell}$ of the type as defined in (2.4) with the corresponding character

defined as in (2.6), but using $y_{-\kappa}$. It is easy to check that the corresponding stabilizer $H_{n-j+1}^{y_{-\kappa}}$ is equal to $U(q_{\eta^{-1}W_j})$, which is isomorphic to G_{n-j} .

Define $C_{\beta-1,\eta} := V_{\beta-1,\eta} \cap V_{\beta,\eta}$, which is also equal to

$$\{u \in V_{\beta-1,\eta} \mid u \cdot e_{\tilde{m}} = e_{\tilde{m}}\}$$

and is a normal subgroup of $H_{n-\ell}^\eta$. It follows that

$$C_{\beta-1,\eta} \backslash H_{n-\ell}^\eta \cong P_\beta^1 \times H_{n-j+1}^{y_{-\kappa}},$$

where P_β^1 is the mirabolic subgroup of $\text{Res}_{E/F}(\text{GL}_\beta)$ given by

$$P_\beta^1 = \left\{ \begin{pmatrix} d & d_1 \\ 0 & 1 \end{pmatrix} \in \text{Res}_{E/F}(\text{GL}_\beta) \right\}.$$

Going back to the expression (3.30) of $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$, the inner integral

$$(3.34) \quad \Phi(h) := \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \phi_\lambda^{\psi_{Z_\ell', \kappa}}(\epsilon_{0,\beta} \eta n h) \psi_{\ell, w_0}^{-1}(n) \, dn$$

as function in h , is left $C_{\beta-1,\eta}(\mathbb{A})$ -invariant. We recall that N_ℓ consists of elements of form

$$\begin{pmatrix} c & x_1 & x_2 & x_3 & y_6 & x_4 & x_5 \\ & I_{\tilde{m}-j} & & & & & x'_4 \\ & & I_{j-\ell} & & & & y'_6 \\ & & & I_{m-2\tilde{m}} & & & x'_3 \\ & & & & I_{j-\ell} & & x'_2 \\ & & & & & I_{\tilde{m}-j} & x'_1 \\ & & & & & & c^* \end{pmatrix}$$

where $c \in Z_\ell$ and the stabilizer N_ℓ^η consists element of form

$$\begin{pmatrix} c & 0 & 0 & 0 & y_6 & 0 & 0 \\ & I_{\tilde{m}-j} & & & & & 0 \\ & & I_{j-\ell} & & & & y'_6 \\ & & & I_{m-2\tilde{m}} & & & 0 \\ & & & & I_{j-\ell} & & 0 \\ & & & & & I_{\tilde{m}-j} & 0 \\ & & & & & & c^* \end{pmatrix}.$$

Then $N_\ell^\eta \backslash N_\ell$ is isomorphic to the subgroup consisting of elements of form

$$\begin{pmatrix} I_\ell & x_1 & x_2 & 0 & x_3 \\ & I_{j-\ell} & & & 0 \\ & & I_{m-2j} & & x'_2 \\ & & & I_{j-\ell} & x'_1 \\ & & & & I_\ell \end{pmatrix}$$

and $\psi_{\ell,\kappa}$ is not trivial on x_2 . In details, $\psi_{\ell,\kappa}|_{N_\ell^\eta \backslash N_\ell} = \psi((x_2)_{\ell,j-\ell})$.

The stabilizer $(\epsilon_{0,\beta}\eta)N_\ell^\eta(\epsilon_{0,\beta}\eta)^{-1}$ in P_j consists of elements of form

$$\begin{pmatrix} I_{j-\ell} & y'_6 & & & \\ & c^* & & & \\ & & e & & \\ & & & c & y_6 \\ & & & & I_{j-\ell} \end{pmatrix}$$

The integral domain $N_\ell^\eta \backslash N_\ell$ under adjoint action of $\epsilon_{0,\beta}\eta$ is a subgroup $U_{j,\eta}^-$ of U_j^- (opposite of the unipotent radical U_j) consisting of elements of form

$$(3.35) \quad \begin{pmatrix} I_{j-\ell} & & & & \\ & I_\ell & & & \\ & x'_2 & I_{m-2j} & & \\ x_1 & x_3 & x_2 & I_\ell & \\ & x'_1 & & & I_{j-\ell} \end{pmatrix}.$$

Denote by $\psi_{(j+1,j)}$ the character over $(\epsilon_{0,\beta}\eta)N_\ell^\eta \backslash N_\ell(\epsilon_{0,\beta}\eta)^{-1}$, given by $\psi_{(j+1,j)}(n) = \psi(n_{j+1,j})$. Indeed, this character is associated to the negative root of the simple root $e_j - e_{j+1}$ corresponding to the maximal parabolic subgroup P_ℓ .

Recall that $\eta^{-1}C_{\beta-1,\eta}\eta$ consists of elements of form

$$\begin{pmatrix} I_\ell & & & & & \\ & I_{\beta-1} & 0 & 0 & u & 0 \\ & & 1 & 0 & 0 & u' \\ & & & I_{m-2j} & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & I_{\beta-1} \\ & & & & & & I_\ell \end{pmatrix}.$$

It follows that $N_{\ell,\beta-1}^\eta = Z_{\ell,\beta-1}^\eta V_{\beta-1,\eta} = Z_\beta C_{\beta-1,\eta}$. As a subgroup of P_j , the stabilizer $(\epsilon_{0,\beta}\eta)N_{\ell,\beta-1}^\eta(\epsilon_{0,\beta}\eta)^{-1}$ consists of elements of form

$$\begin{pmatrix} d & d_1 & 0 & u & 0 & v_1 & v \\ & 1 & & & & & v'_1 \\ & & I_\ell & & & & 0 \\ & & & I_{m-2j} & & & u' \\ & & & & I_\ell & & 0 \\ & & & & & 1 & d'_1 \\ & & & & & & d^* \end{pmatrix},$$

where $d \in Z_{\beta-1}$. Note that $(\epsilon_{0,\beta}\eta)Z_\beta(\epsilon_{0,\beta}\eta)^{-1}$ consists of elements of the above form with all matrices being zero except d and d_1 .

It follows that the expression in (3.30) of the global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ is equal to

$$(3.36) \quad \int_{H_{n-\ell}^\eta(F)C_{\beta-1,\eta}(\mathbb{A}) \backslash H_{n-\ell}(\mathbb{A})} \Phi(h) \int_{[C_{\beta-1,\eta}]} \varphi_\pi(ch) \, dc \, dh,$$

where $[C_{\beta-1,\eta}] := C_{\beta-1,\eta}(F) \backslash C_{\beta-1,\eta}(\mathbb{A})$, as before.

We denote the inner integration \int_c by

$$\varphi_\pi^{C_{\beta-1,\eta}}(h) = \int_{[C_{\beta-1,\eta}]} \varphi_\pi(ch) \, dc.$$

The integral in (3.36) becomes

$$(3.37) \quad \int_{H_{n-\ell}^\eta(F)C_{\beta-1,\eta}(\mathbb{A}) \backslash H_{n-\ell}(\mathbb{A})} \Phi(h) \varphi_\pi^{C_{\beta-1,\eta}}(h) \, dh.$$

Now we are in the standard step in the global unfolding process using partial Fourier expansion along the mirabolic subgroup P_β^1 . Both functions $\Phi(h)$ and $\varphi_\pi^{C_{\beta-1,\eta}}(h)$ are automorphic on $P_\beta^1(\mathbb{A})$ and $\varphi_\pi^{C_{\beta-1,\eta}}(h)$ is cuspidal because of the cuspidality of $\varphi_\pi(h)$. Following the standard Fourier expansion of cuspidal automorphic forms on general linear group ([S74] and [PS79], see also [JL12]), we have

$$(3.38) \quad \varphi_\pi^{C_{\beta-1,\eta}}(h) = \sum_{d \in Z_{\beta-1}(F) \backslash \mathrm{GL}_{\beta-1}(E)} \mathcal{B}^{\psi_{\beta-1,y-\kappa}^{-1}}(\varphi_\pi) \left(\eta^{-1} \begin{pmatrix} I_\ell & & \\ & d & \\ & & 1 \end{pmatrix}^\wedge \eta h \right),$$

which converges absolutely and uniformly in g varying in compact subsets. Recall that the Bessel-Fourier coefficient with respect to $\psi_{\beta-1,y-\kappa}$ is defined as in (2.11) by

$$\mathcal{B}^{\psi_{\beta-1,y-\kappa}^{-1}}(\varphi_\pi)(h) = \int_{N_{\ell,\beta-1}^\eta(F) \backslash N_{\ell,\beta-1}^\eta(\mathbb{A})} \varphi_\pi(nh) \psi_{\beta-1,y-\kappa}(n) \, dn.$$

By using (3.38), the expression (3.37) of $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ is equal to

$$(3.39) \quad \int_{Z_\beta(F)H_{n-j+1}^{y-\kappa}(F)C_{\beta-1, \eta}(\mathbb{A}) \backslash H_{n-\ell}(\mathbb{A})} \Phi(h) \mathcal{B}^{\psi_{\beta-1, y-\kappa}^{-1}}(\varphi_\pi)(h) dh.$$

By pulling out the integration on $Z_\beta(F) \backslash Z_\beta(\mathbb{A})$ and using the fact that $\mathcal{B}^{\psi_{\beta-1, y-\kappa}^{-1}}(\varphi_\pi)(h)$ is left $(Z_\beta(\mathbb{A}), \psi_{\beta-1, y-\kappa}^{-1})$ -quasi-invariant, the integral in (3.39) is equal to

$$(3.40) \quad \int_{H_{n-j+1}^{y-\kappa}(F)N_{\ell, \beta-1}^\eta(\mathbb{A}) \backslash H_{n-\ell}(\mathbb{A})} \mathcal{B}^{\psi_{\beta-1, y-\kappa}^{-1}}(\varphi_\pi)(h) \int_{[Z_\beta]} \Phi(zh) \psi_{\beta-1, y-\kappa}^{-1}(z) dz dh,$$

where $N_{\ell, \beta-1}^\eta = Z_\beta C_{\beta-1, \eta}$ is as before and $[Z_\beta] := Z_\beta(F) \backslash Z_\beta(\mathbb{A})$.

The inner integration

$$\int_{[Z_\beta]} \Phi(zh) \psi_{\beta-1, y-\kappa}^{-1}(z) dz$$

can be calculated more explicitly. By (3.34), it is equal to

$$(3.41) \quad \int_{[Z_\beta]} \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \phi_\lambda^{\psi_{Z_\ell', \kappa}}(\epsilon_{0, \beta} \eta n z h) \psi_{\ell, w_0}^{-1}(n) dn \psi_{\beta-1, y-\kappa}^{-1}(z) dz.$$

The element $(\epsilon_{0, \beta} \eta) z (\epsilon_{0, \beta} \eta)^{-1}$ is given as above. Combining this subgroup with N_ℓ^η , one obtains a subgroup $(\epsilon_{0, \beta} \eta) N_\ell^\eta Z_\beta (\epsilon_{0, \beta} \eta)^{-1}$ of P_j consisting of elements of form

$$(3.42) \quad \begin{pmatrix} d & d_1 & (y'_6)_{*,*} & & & \\ & 1 & (y'_6)_{\beta,*} & & & \\ & & c^* & & & \\ & & & I_{m-2j} & & \\ & & & & c & (y_6)_{*,\beta} & (y_6)_{*,*} \\ & & & & & 1 & d'_1 \\ & & & & & & d \end{pmatrix}.$$

Define

$$\phi^{Z_j, \kappa}(h) = \int_{[Z_\beta]} \phi_\lambda^{\psi_{Z_\ell', \kappa}}(\epsilon_{0, \beta} \eta z h) \psi_{\beta-1, y-\kappa}^{-1} dz.$$

Then,

$$\phi^{Z_j, \kappa}(h) = \int_{[Z_j]} \phi_\lambda(zh) \psi_{Z_j, \kappa} dz,$$

where $\psi_{Z_j, \kappa}(z)$ is given by

$$(3.43) \quad \psi(-z_{1,2} - \cdots - z_{\beta-1, \beta} + (-1)^{m+1} \frac{\kappa}{2} z_{\beta, \beta+1} + z_{\beta+1, \beta+2} + \cdots + z_{j-1, j})$$

with $\beta = j - \ell$. Hence,

$$\begin{aligned} \int_{[Z_\beta]} \Phi(zh) dz &= \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \phi^{Z_{j,\kappa}}(\epsilon_{0,\beta}\eta nh) \psi_{\ell,\kappa}^{-1}(n) dn \\ &= \int_{U_{j,\eta}^-(\mathbb{A})} \phi^{Z_{j,\kappa}}(n\epsilon_{0,\beta}\eta h) \psi_{j+1,j}(n) dn. \end{aligned}$$

Denote the last integral by

$$\mathcal{J}_{\ell,\kappa}(\phi^{Z_{j,\kappa}})(h) = \int_{U_{j,\eta}^-(\mathbb{A})} \phi^{Z_{j,\kappa}}(nh) \psi_{j+1,j}(n) dn.$$

Recall that the group $U_{g,\eta}^-$ consists of elements of form (3.35).

Therefore, we obtain, from (3.36) and (3.37), that the global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell,w_0})$ equals

$$\int_{R_{\ell,\beta-1}^\eta(\mathbb{A}) \backslash H_{n-\ell}(\mathbb{A})} \int_{[H_{n-\ell}^\eta]} \mathcal{B}^{\psi_{\beta-1,y-\kappa}^{-1}}(\varphi_\pi)(xh) \mathcal{J}_{\ell,\kappa}(\phi^{Z_{j,\kappa}})(\epsilon_{0,\beta}\eta xh) dx dh,$$

where $[H_{n-\ell}^\eta] := H_{n-\ell}^\eta(F) \backslash H_{n-\ell}^\eta(\mathbb{A})$.

Proposition 3.7 (Case $(j > \ell)$). *Let $E(\phi_{\tau \otimes \sigma}, s)$ be the Eisenstein series on $G_n(\mathbb{A})$ as in (2.15) and π be an irreducible cuspidal automorphic representation of $H_{n-\ell}(\mathbb{A})$. The global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell,w_0})$ as in (3.1) is equal to*

$$\int_{R_{\ell,\beta-1}^\eta(\mathbb{A}) \backslash H_{n-\ell}(\mathbb{A})} \int_{[H_{n-\ell}^\eta]} \mathcal{B}^{\psi_{\beta-1,y-\kappa}^{-1}}(\varphi_\pi)(xh) \mathcal{J}_{\ell,\kappa}(\phi^{Z_{j,\kappa}})(\epsilon_{0,\beta}\eta xh) dx dh$$

with $[H_{n-\ell}^\eta] := H_{n-\ell}^\eta(F) \backslash H_{n-\ell}^\eta(\mathbb{A})$.

In order to show the integral expression for the global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell,w_0})$ as in Proposition 3.7, it is enough to show that the inner integral

$$(3.44) \quad \int_{[H_{n-\ell}^\eta]} \mathcal{B}^{\psi_{\beta-1,y-\kappa}^{-1}}(\varphi_\pi)(xh) \mathcal{J}_{\ell,\kappa}(\phi^{Z_{j,\kappa}})(\epsilon_{0,\beta}\eta xh) dx$$

is an eulerian product. In fact, for a fixed h , as a function in x , $\mathcal{J}_{\ell,\kappa}(\phi^{Z_{j,\kappa}})(\epsilon_{0,\beta}\eta xh)$ belongs to the space of automorphic representation σ of $G_{n-j}(\mathbb{A})$. Hence, for a fixed h , this above inner integral is essentially the standard Bessel period for the pair (π, σ) as defined in (2.12). By the local uniqueness of the Bessel models ([AGRS10], [SZ12], [JSZ11] and also [GGP12]), integral (3.44) can be written as an eulerian product:

$$(3.45) \quad \prod_{\nu} \langle \mathcal{B}_\nu^{\psi_{\beta-1,y-\kappa_\nu}^{-1}}(\varphi_{\pi,\nu})(h), \mathcal{J}_{\ell,\kappa_\nu}(\phi_{\tau \otimes \sigma,\nu}^{Z_{j,\kappa_\nu}})(\epsilon_{0,\beta}\eta h) \rangle_{G_{n-j}}.$$

Here the local pairing is a linear functional in the Hom-space

$$\mathrm{Hom}_{G_{n-j}(F_\nu)}(\mathcal{B}_\nu^{\psi_{\beta^{-1}, y-\kappa_\nu}^{-1}}(\pi_\nu) \otimes \sigma_\nu, \mathbb{C})$$

with $\mathcal{B}_\nu^{\psi_{\beta^{-1}, y-\kappa_\nu}^{-1}}(\pi_\nu)$ is the local Bessel functional of π_ν . The local uniqueness of the Bessel models ([AGRS10], [SZ12], [JSZ11] and also [GGP12]) asserts that the above Hom-space is at most one-dimensional. One can normalize the local pairing suitable at unramified local places, so that the eulerian product makes sense. Hence we obtain the following theorem.

Theorem 3.8. *Let $E(\phi_{\tau \otimes \sigma}, s)$ be the Eisenstein series on $G_n(\mathbb{A})$ as in (2.15) and let π be an irreducible cuspidal automorphic representation of $H_{n-\ell}(\mathbb{A})$. Assume that the real part of s , $\mathrm{Re}(s)$, is large; and that π and σ have a non-zero Bessel period, i.e. $\mathcal{P}^{\psi_{\beta^{-1}, y-\kappa}^{-1}}(\varphi_\pi, \varphi_\sigma)$ is nonzero for a some choice of data. Then the global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ is eulerian, i.e. is equal to*

$$\prod_v \int_h \langle \mathcal{B}_v^{\psi_{\beta^{-1}, y-\kappa}^{-1}}(\varphi_{\pi, v})(h), \mathcal{J}_{\ell, \kappa}(\phi_{\tau \otimes \sigma, v}^{Z_{j, \kappa}})(\epsilon_{0, \beta} \eta h) \rangle_{G_{n-j}} dh,$$

where the integration is taken over $R_{\ell, \beta-1}^\eta(F_v) \backslash H_{n-\ell}(F_v)$, and the product is taken over all local places.

The main local result of the paper is to calculate the unramified local integral explicitly in terms of the local L -functions. For the purpose of our investigation of the global tensor product L -functions $L(s, \pi \times \tau)$, it is enough to consider the case when $j = \ell + 1$. We define the local zeta integral $\mathcal{Z}_v(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ to the local eulerian v -factor in the product in Theorem 3.8, which is

$$(3.46) \quad \int_h \langle \mathcal{B}_v^{\psi_{\beta^{-1}, y-\kappa}^{-1}}(\varphi_{\pi, v})(h), \mathcal{J}_{\ell, \kappa}(\phi_{\tau \otimes \sigma, v}^{Z_{j, \kappa}})(\epsilon_{0, \beta} \eta h) \rangle_{G_{n-j}} dh,$$

where the integration is taken over $R_{\ell, \beta-1}^\eta(F_v) \backslash H_{n-\ell}(F_v)$.

Theorem 3.9 (L -function for case $j = \ell + 1$). *With all data being unramified, the local unramified zeta integral $\mathcal{Z}_v(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ is equal to the following product*

$$(3.47) \quad \prod_{i=1}^r \frac{L(s + \frac{1}{2}, \tau_{i, v} \otimes \pi_v)}{L(s + 1, \tau_{i, v} \times \sigma_v) L(2s_i + 1, \tau_{i, v}, \mathrm{Asai} \otimes \xi^m)} \\ \times \prod_{1 \leq i < j \leq r} \frac{1}{L(2s + 1, \tau_{i, v} \times \tau_{j, v})} \langle f_\pi, f_\sigma \rangle_{G_{n-j}(F_\nu)},$$

where $\langle f_\pi, f_\sigma \rangle_{G_{n-j}(F_\nu)}$ is independent with s .

This theorem will be proved in Section 4. It is also interesting to understand the local zeta integrals when $j > \ell + 1$. We will come back to this issue in our future consideration.

3.4. Eulerian product: $j \leq \ell$ case. In this section, we consider the case $j \leq \ell < \tilde{m}$. By Proposition 3.6, we only need to consider the representative $\epsilon_{0,0}$ and $\eta_{\epsilon, I_{m-2\ell}}$, where ϵ is defined in (3.15). For simplicity, we denote by $\eta = \eta_{\epsilon, I_{m-2\ell}}$.

By (3.18) and (3.19), we decompose N_ℓ^η as $Z_j N_{j, \ell-j}$, where Z_j is identified as a subgroup of G_n , which is the maximal unipotent subgroup of $\mathrm{GL}(V_j^+)$, and

$$N_{j, \ell-j} = \left\{ \begin{pmatrix} I_j & & & & \\ & b_{\ell-j} & & & \\ & & y_4 & & z_4 \\ & & & I_{m-2\ell} & y'_4 \\ & & & & b^* \\ & & & & & I_j \end{pmatrix} \mid b \in Z_{\ell-j} \right\}.$$

Note that $N_{j, \ell-j}$ is the unipotent subgroup of G_{n-j} as defined in (2.4) and the character $\psi_{\ell, \kappa}$ restricted on $N_{j, \ell-j}$ is the character $\psi_{\ell-j, \kappa}$ of the subgroup $N_{\ell-j}$ (of G_{n-j}) as defined in (2.6), which is denoted by $\psi_{n-j, \ell-j; \kappa}$.

$$(3.48) \quad \mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0}) \\ = \int_{[H_{n-\ell}]} \varphi(h) \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \int_{[N_\ell^\eta]} \lambda \phi(\epsilon_{0,0} \eta u n h) \psi_{\ell, w_0}^{-1}(u n) \, du \, dn \, dh.$$

where $[H_{n-\ell}] := H_{n-\ell}(F) \backslash H_{n-\ell}(\mathbb{A})$ and $[N_\ell^\eta] := N_\ell^\eta(F) \backslash N_\ell^\eta(\mathbb{A})$. The inner integral

$$(3.49) \quad \int_{[N_\ell^\eta]} \lambda \phi(\epsilon_{0,0} \eta u n h) \psi_{\ell, w_0}^{-1}(u) \, du$$

can be written as the following integral

$$\int_{[N_{j, \ell-j}]} \int_{[Z_j]} \lambda \phi(\epsilon_{0,0} \eta c u n h) \psi_{\ell, \kappa}^{-1}(c u) \, dc \, du.$$

Since τ is generic, we have a nonzero Whittaker function

$$\phi_\lambda^{\psi_{Z_j, \kappa}}(h) = \int_{[Z_j]} \lambda \phi(\hat{z} g) \psi_{Z_j, \kappa}(z) \, dz,$$

where $\psi_{Z_j, \kappa}$ is the restriction of $\psi_{\ell, \kappa}$ on Z_j . Hence the inner integral (3.49) can be written as

$$(3.50) \quad \int_{[N_\ell^\eta]} \lambda \phi(\epsilon_{0,0} \eta u n h) \psi_{\ell, w_0}^{-1}(u) du = \mathcal{B}^{\psi_{n-j, \ell-j, \kappa}^{-1}}(\phi_\lambda^{\psi_{Z_j, \kappa}})(\epsilon_{0,0} \eta n h),$$

where $\mathcal{B}^{\psi_{n-j, \ell-j, \kappa}^{-1}}$ is the Bessel period on the group $G_{n-j}(\mathbb{A})$ with respect to the subgroup $N_{j, \ell-j}$ and the character $\psi_{n-j, \ell-j, \kappa}$.

Therefore, the global zeta integral has the expression:

$$\begin{aligned} & \mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0}) \\ &= \int_{[H_{n-\ell}]} \varphi(h) \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \mathcal{B}^{\psi_{n-j, \ell-j, \kappa}^{-1}}(\phi_\lambda^{\psi_{Z_j, \kappa}})(\epsilon_{0,0} \eta n h) \psi_{\ell, w_0}^{-1}(n) dn dh \end{aligned}$$

Proposition 3.10 (Case $(j \leq \ell)$). *Let $E(\phi_{\tau \otimes \sigma}, s)$ be the Eisenstein series on $G_n(\mathbb{A})$ as in (2.15) and π be an irreducible cuspidal automorphic representation of $H_{n-\ell}(\mathbb{A})$. The global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ as in (3.1) is equal to*

$$\int_{[H_{n-\ell}]} \varphi(h) \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \mathcal{B}^{\psi_{n-j, \ell-j, \kappa}^{-1}}(\phi_\lambda^{\psi_{Z_j, \kappa}})(\epsilon_{0,0} \eta n h) \psi_{\ell, w_0}^{-1}(n) dn dh.$$

It remains to show that the global zeta integral in Proposition 3.10 is eulerian. To this end, we need to switch the order of the integrations \int_h and \int_n in

$$\int_{[H_{n-\ell}]} \varphi(h) \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \mathcal{B}^{\psi_{n-j, \ell-j, \kappa}^{-1}}(\phi_\lambda^{\psi_{Z_j, \kappa}})(\epsilon_{0,0} \eta n h) \psi_{\ell, w_0}^{-1}(n) dn dh.$$

This can be deduced from the following lemma.

Lemma 3.11. *The automorphic function*

$$\Psi(h) = \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \mathcal{B}^{\psi_{n-j, \ell-j, \kappa}^{-1}}(\phi_\lambda^{\psi_{Z_j, \kappa}})(\epsilon_{0,0} \eta n h) \psi_{\ell, w_0}^{-1}(n) dn$$

is uniformly moderate growth on $H_{n-\ell}(\mathbb{A})$.

Proof. The proof is similar to the orthogonal case in Appendix 2 to §5 [GPSR97]. \square

Since φ_π is rapidly decay, the global zeta integral is equal to

$$(3.51) \quad \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \int_{[H_{n-\ell}]} \varphi(h) \mathcal{B}^{\psi_{n-j, \ell-j, \kappa}^{-1}}(\phi_\lambda^{\psi_{Z_j, \kappa}})(\epsilon_{0,0} \eta n h) \psi_{\ell, w_0}^{-1}(n) dh dn.$$

Then the inner integral is an $H_{n-\ell}(\mathbb{A})$ invariant pairing between π and $\mathcal{B}_{n-j, \ell-j, \kappa}^{\psi^{-1}}(\sigma)$. The local pairing is a linear functional in the Hom-space

$$\mathrm{Hom}_{G_{n-j}(F_\nu)}(\pi_\nu \otimes \mathcal{B}_\nu^{\psi_{\beta^{-1}, y-\kappa_\nu}^{-1}}(\sigma_\nu), \mathbb{C})$$

with $\mathcal{B}_\nu^{\psi_{\beta^{-1}, y-\kappa_\nu}^{-1}}(\sigma_\nu)$ is the local Bessel functional of σ_ν . By the local uniqueness of the Bessel models and a suitable normalization at unramified local places, we can factorize (3.51) as follows

$$\prod_\nu \left\langle \varphi_\nu * h_\nu, \mathcal{B}_\nu^{\psi_{n-j, \ell-j, \kappa}^{-1}}(\phi_\lambda^{\psi_{Z_j, \kappa}})(\epsilon_{0,0}\eta n_\nu h_\nu) \right\rangle_\nu.$$

Note that in this case, $N_\ell^\eta \backslash N_\ell$ consists of elements of form

$$\begin{pmatrix} I_j & x_1 & x_2 & x_3 & x_4 \\ & I_{\ell-j} & & & x'_3 \\ & & I_{m-2\ell} & & x'_2 \\ & & & I_{\ell-j} & x'_1 \\ & & & & I_j \end{pmatrix}.$$

The restriction of $\psi_{\ell, \kappa}$ on $N_\ell^\eta \backslash N_\ell$ is $\psi((x_1)_{j,1})$. Under the adjoint action of $\epsilon_{0,0}\eta$, the integral domain $N_\ell^\eta \backslash N_\ell$ is also denoted by $U_{j,\eta}^-$, which is a subgroup of the opposite U_j^- , consisting of elements of form

$$\begin{pmatrix} I_j & & & & \\ x'_3 & I_{\ell-j} & & & \\ x'_2 & & I_{m-2\ell} & & \\ x'_1 & & & I_{\ell-j} & \\ x_4 & x_1 & x_2 & x_3 & I_j \end{pmatrix}$$

Therefore, the induced character on $U_{j,\ell}^{-1}$ is $\psi^{-1}(n_{m-j,1})$.

We remark that in the family of global zeta integrals, we only use the case when $j = \ell + 1$ to calculate the local unramified zeta integrals to obtain the local L -functions as we needed. In all other cases, the global zeta integrals are eulerian. The unramified calculation will be taken up in our future consideration, and the potential applications to our explicit constructions of endoscopy correspondences as discussed in [J12] remain to be fully discussed.

4. UNRAMIFIED CALCULATION AND LOCAL L -FUNCTIONS

In this section, we will calculate the local zeta integral for the case $j = \ell + 1$ as defined in Theorem 3.9 over the unramified places. The

quasi-split orthogonal group cases were done in [GPSR97]. In the following, we extend the idea and the method in [GPSR97] to the quasi-split unitary group cases. It turns out that the argument in this case is much more technically involved, due to the splitting of the unramified local place of the number field F to the quadratic extension E .

To achieve the goal of this section, we reformulate the local zeta integrals through the paring of Bessel models in Subsection 4.1, including some general statements on the twisted Jacquet modules, which we recall from [GRS11, Chapter 5]. In Subsection 4.2, we discuss unramified representations considered in the local zeta integrals and their Satake parameters, with which, we define unramified local L -functions we need. In Subsection 4.3, we specify the local zeta integrals for unramified data by considering the cases when the unramified local place ν of F is split or not over E . By the Bernstein rationality, the unramified local zeta integrals are expressed as a rational function with respect to the parameters coming from the relevant representations. This rational function is explicitly calculated in Subsections 4.4 and 4.5 and identified with the expected local L -functions. Hence we carry out the complete proof of Theorem 3.9.

Throughout this section, denote by ν the local place of F . If ν is inert, then E_ν is the unramified quadratic extension of F_ν . If ν splits over E , then $E_\nu \cong F_\nu \times F_\nu$. For the simplicity of notation, most of the time, we will omit the subscript ν from the corresponding notation. For instance, we may use F for the local field F_ν and use π for π_ν and so on, when there are no confusions.

Let \mathfrak{o} be the ring of integers of F , and fix a prime element ϖ of \mathfrak{o} . Let q_F and q_E be the cardinality of the residue fields of F and E , respectively. If ν is inert, one has that $q_E = q_F^2$, and if ν is split, one has that $q_E = q_F$. We fix the normalized absolute values $|x|_F = |x|_\nu$ for $x \in F$, and $|x|_E = |x\bar{x}|_F$ for $x \in E$ if ν is inert.

When ν splits in E , we need to write down more explicit structures of the unitary group $G_n(F)$, which are needed for the unramified calculation of the local integrals. In this case, one may take that $\rho = d^2$ or $\sqrt{\rho} = d$ for some $d \in F^\times$, and hence has that $E \cong F \times F$. This isomorphism is explicitly given by the following mapping: for $x, y \in F$,

$$x \otimes 1 + y \otimes \sqrt{\rho} \mapsto (x + yd, x - yd).$$

When $x \in E$ is taken to $(x_1, x_2) \in F \times F$, the corresponding absolute values are normalized so that $|x|_E = |x_1 x_2|_F$. It follows that $\mathrm{GL}_m(E) \cong \mathrm{GL}_m(F) \times \mathrm{GL}_m(F)$ given by

$$g_1 \otimes 1 + g_2 \otimes \sqrt{\rho} \mapsto (g_1 + dg_2, g_1 - dg_2).$$

Then the unitary group $G_n(F)$ consists of all elements $g = g_1 \otimes 1 + g_2 \otimes \sqrt{\rho} \in \mathrm{GL}_m(E)$ satisfying

$$(g_1 + dg_2)J_m^t(g_1 - dg_2) = J_m.$$

The restriction of the above isomorphism to $G_n(F)$ gives the isomorphism: $G_n(F) \cong \mathrm{GL}_m(F)$, given explicitly by

$$(4.1) \quad g_1 \otimes 1 + g_2 \otimes \sqrt{\rho} \mapsto (g_1 + dg_2, g_1 - dg_2) \mapsto g_1 + dg_2.$$

Next, we explain the data in the unramified local integral as needed for Theorem 3.9. We take a normalized parabolically induced representation

$$\Pi(\tau, \sigma, s) = \mathrm{Ind}_{P_j(F)}^{G_n(F)} (|\det|_E^s \tau \otimes \sigma),$$

where τ and σ are irreducible admissible representations of $\mathrm{GL}_j(E)$ and $G_{n-j}(F)$, respectively. Let π be an irreducible admissible representation of $H_{n-\ell}(F)$. Recall that the unitary group $H_{n-\ell}$ is defined in (2.9).

When ν splits in E , the induced representation $\Pi(\tau, \sigma, s)$ can be made more specific. In this case, the representation τ can be expressed as $\tau_1 \otimes \tau_2$, where τ_i are irreducible representations of $\mathrm{GL}_j(F)$. The representation σ is an irreducible representation of $\mathrm{GL}_{m-2j}(F)$. The representation $\Pi(\tau, \sigma, s)$ can be realized as the representation of $\mathrm{GL}_m(F)$, induced from the standard parabolic subgroup $P_{j,m-2j,j}(F)$ with the following representation

$$\begin{pmatrix} g_1 & x & y \\ & h & z \\ & & g_2 \end{pmatrix} \mapsto \left| \frac{\det(g_1)}{\det(g_2)} \right|^s \tau_1(g_1) \otimes \sigma(h) \otimes \tau_2(g_2^*),$$

where $g_1, g_2 \in \mathrm{GL}_j(F)$ and $g_2^* = J_j^t g^{-1} J_j^{-1}$.

4.1. Local zeta integrals and twisted Jacquet modules. We introduce a local zeta integral in general at any local place, although only a special case will contribute to the proof of Theorem 3.9.

Let W_j be a Whittaker model attached to a nonzero member in the space

$$\mathrm{Hom}_{\mathrm{GL}_j(F)}(\tau, \mathrm{Ind}_{Z_j(E)}^{\mathrm{GL}_j(E)}(\psi_{Z_j, \kappa})).$$

This produces a partial Whittaker function

$$W_j(f) \in \mathrm{Ind}_{Z_j(E) \times G_{n-j}(F) \ltimes U_j(F)}^{G_n(F)}(\psi_{Z_j, \kappa} \otimes \sigma)$$

for $f \in \Pi(\tau, \sigma, s)$. As suggested by the global calculation in Section 3, we can formally define (over the open cell) the following function

$$\mathcal{J}(f)(g) := \int_{N_\ell^\eta(F) \backslash U_\ell(F)} W_j(f)(\epsilon_{0,j-\ell} \eta u g) \psi_{(m-j+\ell, j-\ell)}^{-1}(u) du.$$

Following the exact argument in Appendix 2 to §5 [GPSR97], the integral defining $\mathcal{J}(f)$ is convergent for $\text{Re}(s)$ sufficient large and is analytic in s . In addition, $\mathcal{J}(f)$ is a function on $H_{n-\ell}(F)$ belonging to the space

$$\text{Ind}_{R_{\ell, \beta-1}^\eta(F)}^{H_{n-\ell}(F)}(\psi_{\beta-1, y-\kappa}^{-1} \otimes \sigma^{w_q^\ell}),$$

where $\sigma^{w_q^\ell}$ is a representation of $G_{n-j}(F)$ conjugated by w_q^ℓ .

Let $\mathcal{B}_{\beta-1}$ be a Bessel model attached to a non-trivial member in the Hom-space

$$\text{Hom}_{H_{n-\ell}(F)}(\pi, \text{Ind}_{R_{\ell, \beta-1}^\eta(F)}^{H_{n-\ell}(F)}(\psi_{\beta-1, y-\kappa} \otimes \tilde{\sigma}^{w_q^\ell})),$$

where $\tilde{\sigma}$ is the dual of σ . Let $\langle \cdot, \cdot \rangle_\sigma$ be an invariant pairing of σ and $\tilde{\sigma}$. By the uniqueness of the local Bessel models ([AGRS10], [GGP12], [SZ12] and [JSZ11]), $\mathcal{B}_{\beta-1}$ is unique up to a constant. We may define a pairing

$$\langle \mathcal{J}(f), \mathcal{B}_{\beta-1}(v) \rangle = \int_{R_{\ell, \beta-1}^\eta(F) \backslash H_{n-\ell}(F)} \langle \mathcal{J}(f)(h), \mathcal{B}_{\beta-1}(v)(h) \rangle_\sigma dh.$$

Lemma 4.1. *The pairing $\langle \mathcal{J}(f), \mathcal{B}_{\beta-1}(v) \rangle$ is absolutely convergent for suitable data τ and σ , and $\text{Re}(s)$ sufficiently large.*

Proof. The proof is similar to Theorem A of Appendix (I) to §5 in [GPSR97]. \square

It is easy to check that this pairing, if exists, defines a linear functional of the Gross-Prasad type in the Hom-space

$$(4.2) \quad \text{Hom}_{N_\ell \times H_{n-\ell}^\Delta}(\Pi(\tau, \sigma, s) \otimes \pi, \psi_{\ell, \kappa}).$$

Again, by the uniqueness of local Bessel functionals, the dimension of this Hom-space is at most one, when $\Pi(\tau, \sigma, s)$ is irreducible. Therefore, the local zeta integral is defined by

$$(4.3) \quad \mathcal{Z}(s, f, v, \psi_{\ell, \kappa}) = \langle \mathcal{J}(f), \mathcal{B}_{\beta-1}(v) \rangle,$$

for $f \in \Pi(\tau, \sigma, s)$ and $v \in \pi$, which is proportional to the local zeta integral defined as an eulerian factor of the the global zeta integral as in Section 3. For the unramified data, we may normalize the pairing, so that this proportional constant is one.

In order to proceed the explicit calculation of the local integrals, we have to understand those Bessel models involved in the local zeta integrals from the representation-theoretic point of view. This means to see more precisely the structures of those twisted Jacquet models. We recall relevant results from [GRS11, Chapter 5].

Let (Π, V_Π) be a smooth representation of $G_n(F)$. Let $J_{\psi_{\ell,\kappa}}(\Pi)$ be the twisted Jacquet module of Π with respect to $N_\ell(F)$ and its character $\psi_{\ell,\kappa}$, the space of which is defined by

$$(4.4) \quad V_\Pi / \text{Span} \{ \Pi(n)v - \psi_{\ell,\kappa}(n)v \mid n \in N_\ell(F), v \in V_\Pi \}.$$

Note that $J_{\psi_{\ell,\kappa}}(\Pi)$ is a smooth representation of $H_{n-\ell}(F)$. The same definition may apply to the twisted Jacquet modules for different groups throughout the section.

Next, we study the twisted Jacquet module $J_{\psi_{\ell,\kappa}}(\Pi)$ for the induced representation $\Pi = \Pi(\tau, \sigma, s)$. To do so, we consider the structure of the restriction of the induced representation Π to the standard parabolic subgroup P_ℓ , which is denoted by $\text{Res}_{P_\ell}(\Pi)$. This reduces to consider the generalized Bruhat decomposition $P_j \backslash G_n / P_\ell$, which was discussed in Section 3. Hence, as a representation of P_ℓ , $\text{Res}_{P_\ell}(\Pi)$ can be expressed (up to semi-simplification) as a finite direct sum $\Pi_{\epsilon_{\alpha,\beta}}$ parameterized by the set of representatives $\{\epsilon_{\alpha,\beta}\}$ as discussed in Section 3.1.

Let $\tau^{(t)}$ denote the t -th Bernstein-Zelevinski derivative of τ along the subgroup Z'_t defined in (3.26) with the character

$$\psi'_t \begin{pmatrix} I_\beta & y \\ 0 & z \end{pmatrix} = \psi^{-1}(z_{1,2} + z_{2,3} + \cdots + z_{t-1,t}).$$

We embed GL_β into GL_j through the map $g \in \text{GL}_\beta \mapsto \text{diag}(g, I_t) \in \text{GL}_j$. The image, which is still denoted by GL_β , normalizes the character ψ'_t . Hence $\tau^{(t)}$ is the representation of GL_β via the twisted Jacquet module $J_{\psi'_t}(\tau)$. We also define the following character of Z'_t ,

$$\psi''_t \begin{pmatrix} I_\beta & y \\ 0 & z \end{pmatrix} = \psi^{-1}(z_{1,2} + z_{2,3} + \cdots + z_{t-1,t} + y_{\beta,1}),$$

which is conjugate to the character $\psi_{Z'_t,\kappa}$ as defined in (3.28) for any nonzero κ , by an element in the subgroup GL_β . Denote the corresponding Jacquet module $J_{\psi''_t}(\tau)$ by $\tau_{(t)}$, which is a representation of the mirabolic subgroup of GL_β .

Recall that $P'_\beta = H_{n-\ell}^{\eta_{\epsilon,I_{m-2\ell}}}$ is as defined in (3.9). By the discussion in Page 11, when $\ell + \beta < \tilde{m}$, P'_β is a maximal parabolic subgroup of $H_{n-\ell}$. For the proof of Theorem 3.9, which only concerns the case of $j = \ell + 1$, we may assume that $\ell < j$ in the following discussion. Put $P''_{j-\ell} = H_{n-\ell}^{\eta_{\epsilon,\gamma}}$ for γ as defined in (3.16). Note that $P'_w \gamma H_{n-\ell}$ is the open double coset discussed in Page 18, and $P''_{j-\ell}$ is not a proper maximal parabolic subgroup. Although we only need in this paper the case when $\ell < j$, we recall from [GRS11] the following general result.

Proposition 4.2 ([GRS11, Theorem 5.1]). *Assume that $0 \leq \ell < \tilde{m}$ and $1 \leq j < m$. If ν is inert, then, up to semi-simplification, the following isomorphism holds*

$$J_{\psi_{\ell,\kappa}}(\text{Ind}_{P_j}^{G_n}(\tau \otimes \sigma)) \equiv \Upsilon_1 \oplus \Upsilon_2 \oplus \Upsilon_3$$

where

$$\begin{aligned} \Upsilon_1 &= \bigoplus_{\substack{j-\ell \leq \beta < \tilde{m}-\ell \\ 0 \leq \beta \leq j}} \text{ind}_{P'_\beta}^{H_{n-\ell}}(|\det|_E^{\frac{1-t}{2}} \tau^{(t)} \otimes J_{\psi'_{\ell-t,\kappa}}(\sigma^{w_q^t})), \\ \Upsilon_2 &= \begin{cases} \text{ind}_{P''_{j-\ell}}^{H_{n-\ell}}(|\det|_E^{-\frac{\ell}{2}} \tau_{(\ell)} \otimes \sigma^{w_q^\ell}), & \ell < j, \\ 0, & \ell \not< j, \end{cases} \end{aligned}$$

and Υ_3 is the representation of $H_{n-\ell}$ supported on the other double cosets.

We note that the detail of Υ_3 is not needed in the explicit unramified calculation and is referred to [GRS11, Theorem 5.1].

If ν is split, let $\underline{\ell} = [\ell_1, \ell_2, \ell_3]$ be a partition of a positive integer N and consider the twisted Jacquet module $J_{\tilde{\psi}}(\text{Ind}_{P_{j,N-j}}^{\text{GL}_N} \tau_1 \times \tau_2)$ in [GRS11, Section 3.6]. In order to simplify our calculation, up to a suitable conjugation, we will use the Gelfand-Grave character defined in [GRS11, Section 3.6]. Let N_ℓ consist of elements of form

$$n = \begin{pmatrix} z^{(1)} & y^{(1)} & x \\ & I_{m-2\ell} & y^{(2)} \\ & & z^{(2)} \end{pmatrix} \in \text{GL}_m(F),$$

where $z^{(1)}, z^{(2)} \in Z_\ell(F)$. We will take the character $\psi_{\ell,\kappa}$ to be the following character

$$\tilde{\psi}(n) = \psi\left(\sum_{i=1}^{\ell-1} (z_{i,i+1}^{(1)} + z_{i,i+1}^{(2)}) + y_{\ell,1}^{(1)} + y_{1,1}^{(2)}\right).$$

The stabilizer of the character $\tilde{\psi}(n)$ inside $G_{n-\ell}(E_\nu) \cong \text{GL}_{m-2\ell}(F)$ is

$$\tilde{L}_\ell = \left\{ \text{diag}\{I_\ell, \gamma, I_\ell\} \in \text{GL}_m(F) \mid \gamma = \begin{pmatrix} 1 & \\ & g \end{pmatrix}, g \in \text{GL}_{m-2\ell-1}(F) \right\}.$$

Define

$$\tau_2^{[\ell_1-\alpha]} := [(\tau_2^*)^{\ell_1-\alpha}]^* \text{ and } (\tau_2)_{[\ell_1]} := [(\tau_2^*)_{(\ell_1)}]^*.$$

which are representations of $\text{GL}_{\ell_2-\beta+\alpha}(F)$ and the mirabolic subgroup of $\text{GL}_{N-j-\ell_1}(F)$, respectively, where the inner $*$ denotes composition with the map

$$g \rightarrow J'_{\ell_1-\alpha} {}^t g^{-1} J'^{-1}_{\ell_1-\alpha},$$

where $J'_{\ell_1-\alpha} = \text{diag}(J_{\ell_1-\alpha}, J_{\ell_2-\beta+\alpha})$, and the outer $*$ denotes composition with the map

$$g \rightarrow J_{\ell_2-\beta+\alpha} \cdot {}^t g^{-1} J_{\ell_2-\beta+\alpha}^{-1}.$$

More information about $\tau_2^{[\ell_1-\alpha]}$ and $(\tau_2)_{[\ell_1]}$ can be found in [GRS11, Pages 113 and 115].

Proposition 4.3 ([GRS11, Theorem 5.7]). *Up to semi-simplification, the following isomorphism holds*

$$J_{\tilde{\psi}}(\text{Ind}_{P_{j,N-j}}^{\text{GL}_N(F)} \tau_1 \times \tau_2) \equiv \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4 \oplus \mathcal{L}_5$$

where \mathcal{L}_1 is given by the following direct sum

$$\bigoplus_{\substack{j-\ell_3 < \beta < \ell_2 \\ 0 \leq \beta \leq j}} \text{Ind}_{P_{\beta, \ell_2-\beta-1}}^{\text{GL}_{\ell_2-1}} (|\cdot|^{-\frac{1-(j-\beta)+\ell_3-\ell_1}{2}} \tau_1^{(j-\beta)}) \otimes |\cdot|^{-\frac{j-\beta}{2}} J_{\psi_{(\ell_1, \ell_2-\beta, \ell_3-j+\beta)}}(\tau_2);$$

\mathcal{L}_2 is given by the following direct sum

$$\bigoplus_{\substack{0 < r < j-\ell_3 \\ j-\ell_2-\ell_3 \leq r \leq \ell_1}} \text{Ind}_{P_{j\ell_3-r-1, \ell_2+\ell_3-j+r}}^{\text{GL}_{\ell_2-1}} (|\cdot|^{-\frac{\ell_1-r}{2}} J_{\psi_{(r, j-\ell_3-r, \ell_3)}}(\tau_1)) \otimes |\cdot|^{-\frac{\ell_3-r-1}{2}} \tau_2^{[\ell_1-r]};$$

\mathcal{L}_3 is given by the following representation

$$\begin{cases} \text{Ind}_{P_{j-\ell_3-1, \ell_2+\ell_3-j}}^{\text{GL}_{\ell_2-1}} (|\cdot|^{-\frac{\ell_1}{2}} (\tau_1)_{(\ell_3)}) \otimes |\cdot|^{-\frac{\ell_3-1}{2}} \tau_2^{[\ell_1]}, & \text{if } 0 < j - \ell_3 \leq \ell_2, \\ \tau_1^{(j)} \otimes |\det|^{-\frac{\ell_3}{2}} \tau_2^{[\ell_1]}, & \text{if } \ell_3 = j, \\ 0, & \text{otherwise;} \end{cases}$$

\mathcal{L}_4 is given by the following representation

$$\begin{cases} \text{Ind}_{P_{j-\ell_3, \ell_2+\ell_3-j-1}}^{\text{GL}_{\ell_2-1}} (|\cdot|^{-\frac{1-\ell_1}{2}} (\tau_1)^{(\ell_3)}) \otimes |\cdot|^{-\frac{\ell_3}{2}} \tau_2^{[\ell_1]}, & \text{if } 0 < j - \ell_3 < \ell_2, \\ 0, & \text{otherwise;} \end{cases}$$

and \mathcal{L}_5 is given by the following representation

$$\begin{cases} \text{ind}_{P'_{j-\ell_3-1, 1, \ell_2+\ell_3-j-1}}^{\text{GL}_{\ell_2-1}} (|\cdot|^{-\frac{\ell_1}{2}} (\tau_1)_{(\ell_3)}) \otimes |\cdot|^{-\frac{\ell_3}{2}} \tau_2^{[\ell_1]}, & \text{if } 0 < j - \ell_3 < \ell_2 \\ 0, & \text{otherwise.} \end{cases}$$

The other notation in this proposition is referred to [GRS11, Section 5.2]. We are going to apply the case of $\underline{\ell} = [\ell, m-2j, \ell]$ to the unramified calculation.

4.2. Unramified representations and local L -functions of unitary groups. Let $B_H = T_H N_H$ be a Borel subgroup of $H_{n-\ell}$ with the maximal F -torus T_H and the unipotent radical N_H . Let $K_G = G_n(\mathfrak{o}_F)$ (resp. $K_H = H_{n-\ell}(\mathfrak{o}_F)$) be the standard maximal open compact subgroup of G_n (resp. $H_{n-\ell}$). Denote by $W(G_n) = N(T)/T$ the Weyl

group of G_n . When ν is inert over E , $W(G_n)$ is the Weyl group associated to a root system of type B . When ν is split over E , $W(G_n)$ is the Weyl group associated to a root system of type A .

From now on, we assume that the representations τ , σ , and π are unramified. Let χ_τ and χ_σ be the unramified characters corresponding to the spherical representations τ and σ . Then $\chi_\tau = \otimes_{i=1}^j \chi_i$ and $\chi_\sigma = \otimes_{i=j+1}^{\tilde{m}} \chi_i$. Define $\chi_s := |\cdot|^s \chi_\tau \otimes \chi_\sigma$. Let $\Pi_s := \Pi(\chi_s)$ and $\pi := \pi(\mu)$ be the unramified constituents of the normalized induced representations

$$\text{Ind}_{P_j(F)}^{G_n(F)}(|\det|^s \tau \otimes \sigma) \quad \text{and} \quad \text{Ind}_{B_H(F)}^{H_{n-\ell}(F)}(\mu),$$

respectively.

If ν is inert over E , χ_i and μ_i are unramified characters of $E^\times = F(\sqrt{\rho})^\times$.

If ν is split over E , $H_{n-\ell}(F) \cong \text{GL}_{m-2\ell-1}(F)$ and μ_i splits into a product $\theta_i \vartheta_i$ of two unramified characters of F^\times . Moreover, if $m-2\ell-1$ is odd, μ splits as $\otimes_{i=1}^{(m-2\ell-2)/2} \theta_i \otimes \vartheta_i \otimes \mu_0$. Here μ_0 is also an unramified character of F^\times . In particular, $\pi(\mu)$ is the unramified constituent of the following induced representation

$$\text{Ind}_{B_H}^{H_{n-\ell}(F)}((\otimes_{i=1}^{\tilde{m}_H} \theta_i) \otimes (\otimes_{i=1}^{\tilde{m}_H} \vartheta_{\tilde{m}_H+1-i}^{-1}))$$

if m is odd, and of the following induced representation

$$\text{Ind}_{B_H}^{H_{n-\ell}(F)}((\otimes_{i=1}^{\tilde{m}_H} \theta_i) \otimes \mu_0 \otimes (\otimes_{i=1}^{\tilde{m}_H} \vartheta_{\tilde{m}_H+1-i}^{-1}))$$

if m is even, where \tilde{m}_H is the Witt index of the hermitian vector subspace $(W_\ell \cap w_0^\perp, q_{W_\ell \cap w_0^\perp})$, which defines $H_{n-\ell}$. Since $E \cong F \times F$, we must have

$$\text{GL}_j(E) \cong \text{GL}_j(F) \times \text{GL}_j(F)$$

and χ_τ splits as a product $\Xi_\tau \Theta_\tau$ of unramified characters with

$$\begin{aligned} \Xi_\tau &= \otimes_{i=1}^j \Xi_i, \\ \Theta_\tau &= \otimes_{i=1}^j \Theta_i. \end{aligned}$$

The representation τ is the unramified constituent of the induced representation

$$\text{Ind}_{B_{\text{GL}_j}(F)}^{\text{GL}_j(F)}(|\det|^s \Theta_\tau) \otimes \text{Ind}_{B_{\text{GL}_j}(F)}^{\text{GL}_j(F)}(|\det|^{-s} \Xi_\tau^{-1}).$$

Also, if we set $\chi_i = |\cdot|^s \Theta_i$ and $\chi_{m+1-i} = |\cdot|^{-s} \Xi_i^{-1}$ for $1 \leq i \leq j$, then the representation $\Pi(\chi_s)$ of $G_n(F)$ becomes the corresponding representation of $\text{GL}_m(F)$.

In the following, we write down the Satake parameters for the unramified representations discussed above and write the relevant unramified

local L -functions, following the arguments in [BS09] or [KK11] for instance.

The Langlands dual group ${}^L\mathbf{U}_m$ of \mathbf{U}_m is $\mathrm{GL}_m(\mathbb{C}) \rtimes \Gamma(E/F)$, where $\Gamma(E/F)$ is the Galois group on E and the nontrivial element ι acts on $\mathrm{GL}_m(\mathbb{C})$ via $\iota(g) = J_m {}^t g^{-1} J_m^{-1}$. A $2m$ -dimensional complex representation ρ_{2m} of the Langlands dual group ${}^L\mathbf{U}_m$ is given by

$$(g; 1) \mapsto \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} \text{ and } (I_m; \iota) \mapsto \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix},$$

for any $g \in \mathrm{GL}_m(\mathbb{C})$. The Langlands dual group ${}^L\mathrm{Res}_{E/F}\mathrm{GL}_j$ of $\mathrm{Res}_{E/F}\mathrm{GL}_j$ is $(\mathrm{GL}_j(\mathbb{C}) \times \mathrm{GL}_j(\mathbb{C})) \rtimes \Gamma(E/F)$. The element ι acts on $\mathrm{GL}_j(\mathbb{C}) \times \mathrm{GL}_j(\mathbb{C})$ by $\iota(g_1, g_2) = (g_2, g_1)$. Considering a j^2 dimensional representation of ${}^L\mathrm{Res}_{E/F}\mathrm{GL}_j$, which is realized in the space of all $j \times j$ matrices, $M_{j \times j}$, by

$$\begin{aligned} (g_1, g_2; 1)(x) &\mapsto g_1 \cdot x \cdot {}^t g_2, \\ (I_j, I_j; \iota)(x) &\mapsto {}^t x, \end{aligned}$$

and is called the *Asai* representation of ${}^L\mathrm{Res}_{E/F}\mathrm{GL}_j$.

In addition, the Langlands dual group ${}^L(\mathbf{U}_m \times \mathrm{Res}_{E/F}\mathrm{GL}_j)$ is

$$(\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_j(\mathbb{C}) \times \mathrm{GL}_j(\mathbb{C})) \rtimes \Gamma(E/F).$$

The element ι acts on it by $\iota(g, g_1, g_2) = (g^*, g_2, g_1)$. A $2mj$ -dimensional complex representation ρ_{2mj} of ${}^L(\mathbf{U}_m \times \mathrm{Res}_{E/F}\mathrm{GL}_j)$ is given by

$$\begin{aligned} (g, g_1, g_2, 1) &\mapsto \begin{pmatrix} g \otimes g_1 & 0 \\ 0 & g^* \otimes g_2 \end{pmatrix}, \\ (I_m, I_j, I_j, \iota) &\mapsto \begin{pmatrix} 0 & I_{mj} \\ I_{mj} & 0 \end{pmatrix}, \end{aligned}$$

where $g \otimes g_i$ is the Kronecker product.

We first consider the irreducible unramified representation $\pi(\mu)$ of $H_{n-\ell}(F)$. When ν is inert over E , the Satake parameter of $\pi(\mu)$ is the semi-simple conjugacy class in ${}^LH_{n-\ell}$ of type

$$c(\pi(\mu)) = (\mathrm{diag}(\mu_1(\varpi_E), \mu_2(\varpi_E), \dots, \mu_{\tilde{m}_H}(\varpi_E), 1, \dots, 1); \iota),$$

where ϖ_E is the ν -uniformizer of E . To simplify the notation, we may use μ_i for $\mu_i(\varpi_E)$ in the following, if it does not cause any confusion.

When ν is split over E , the Satake parameter of $\pi(\mu)$ is the semi-simple conjugacy class in ${}^LH_{n-\ell}$ of type

$$c(\pi(\mu)) = (\mathrm{diag}(\theta_1(\varpi), \dots, \theta_{\tilde{m}_H}(\varpi), \vartheta_1^{-1}(\varpi), \dots, \vartheta_{\tilde{m}_H}^{-1}(\varpi)); 1)$$

if m is odd, and of type

$$c(\pi(\mu)) = (\mathrm{diag}(\theta_1(\varpi), \dots, \theta_{\tilde{m}_H}(\varpi), \mu_0(\varpi), \vartheta_1^{-1}(\varpi), \dots, \vartheta_{\tilde{m}_H}^{-1}(\varpi)); 1)$$

if m is even, where ϖ is the ν -uniformizer of F .

Next, we consider the irreducible unramified representation $\tau(\chi_\tau)$ of $\text{Res}_{E/F}(\text{GL}_j)(F)$, where $\chi_\tau = \otimes_{i=1}^j \chi_i$. When ν is inert over E , the Satake parameter of $\tau(\chi_\tau)$ is the semi-simple conjugacy class in ${}^L\text{Res}_{E/F}(\text{GL}_j)$ of type

$$c(\tau(\chi_\tau)) = (\text{diag}(\chi_1(\varpi_E), \chi_2(\varpi_E), \dots, \chi_j(\varpi_E)), I_j; \iota).$$

Again, we use χ_i for $\chi_i(\varpi_E)$ if it does not cause any confusion.

When ν is split over E , the Satake parameter of $\tau(\chi_\tau)$ is the semi-simple conjugacy class in ${}^L\text{Res}_{E/F}(\text{GL}_j)$ of type

$$c(\tau(\chi_\tau)) = (\text{diag}(\Theta_1, \dots, \Theta_j), \text{diag}(\Xi_1, \dots, \Xi_j); 1),$$

where Θ_i is used for $\Theta_i(\varpi)$ and Ξ_i is used for $\Xi_{i,\nu}(\varpi)$, to simplify the notation.

Therefore, if ν is inert over E and E is the unramified quadratic field extension of F , the unramified tensor product local L -function $L(s, \pi \times \tau)$ is defined to be

$$(4.5) \quad \prod_{\substack{1 \leq i \leq j \\ 1 \leq i' \leq \tilde{m}_H}} (1 - \chi_i \mu_{i'} q_F^{-2s})^{-1} (1 - \chi_i \mu_{i'}^{-1} q_F^{-2s})^{-1} \prod_{1 \leq k \leq n} (1 - \chi_k q_F^{-2s})^{-1},$$

if m is even; and to be

$$(4.6) \quad \prod_{\substack{1 \leq i \leq j \\ 1 \leq i' \leq \tilde{m}_H}} (1 - \chi_i \mu_{i'} q_F^{-2s})^{-1} (1 - \chi_i \mu_{i'}^{-1} q_F^{-2s})^{-1},$$

if m is odd. When ν is split over E , the unramified tensor product local L -function $L(s, \pi \times \tau)$ is defined to be

$$(4.7) \quad L(s, \pi \times \tau) = L(s, \pi \times \tau_1) L(s, \tilde{\pi} \times \tau_2),$$

where τ_1 and τ_2 are defined according to $\Theta_1, \dots, \Theta_j$ and Ξ_1, \dots, Ξ_j , respectively.

Moreover, the unramified local *Asai* L -function of τ is defined as, when ν is inert,

$$(4.8) \quad L(s, \tau, \text{Asai}) = \prod_{1 \leq i_1 < i_2 \leq j} (1 - \mu_{i_1} \mu_{i_2} q_F^{-2s})^{-1} \prod_{1 \leq i \leq j} (1 - \mu_i q_F^{-s})^{-1};$$

and when ν is split

$$(4.9) \quad L(s, \tau, \text{Asai}) = L(s, \tau_1 \times \tau_2) = \prod_{1 \leq i, k \leq j} (1 - \Theta_i \Xi_k q_F^{-s})^{-1}.$$

In the same way we define the unramified tensor product local L -function $L(s, \sigma \times \tau)$.

4.3. Unramified local zeta integrals. Let f_{χ_s} and f_{μ} be the spherical functions in $\Pi(\chi_s)$ and $\pi(\mu)$, normalized by $f_{\chi}(e_G) = f_{\mu}(e_H) = 1$. Denote by f_{τ} and f_{σ} the unramified function in τ and σ accordingly. We are going to calculate explicitly the unramified local zeta integral $\mathcal{Z}_{\nu}(s, f_{\chi_s}, f_{\mu}, \psi_{\ell, \kappa})$.

By Bernstein rationality theorem ([GPSR87] and see also [Bn98]), $\mathcal{Z}_{\nu}(s, f_{\chi_s}, f_{\mu}, \psi_{\ell, \kappa})$ is a rational function of the parameters χ_s and μ . Thus, we can assume that

$$(4.10) \quad \mathcal{Z}_{\nu}(s, f_{\chi_s}, f_{\mu}, \psi_{\ell, \kappa}) = \frac{P(\chi_s, \mu)}{Q(\chi_s, \mu)},$$

where $P(\chi_s, \mu)$ and $Q(\chi_s, \mu)$ are polynomials of variables in χ_i , μ_i and q_E^{-s} . We are going to calculate the polynomials $P(\chi_s, \mu)$ and $Q(\chi_s, \mu)$ explicitly in the following two subsections.

4.4. Calculation of $Q(\chi_s, \mu)$. For a technical reason, which will be mentioned in the argument below, we assume that $j = \ell + 1$. This is enough to produce the unramified local L -functions as needed. The method used here is an extension of that in [GPSR97] to the unitary group case. The idea to calculate $Q(\chi_s, \mu)$ is to find a proper Hecke algebra element Φ_0 in the extended spherical Hecke algebra of $H_{n-\ell}$ as defined below, so that for any section f_{χ_s} in the unramified induced representation

$$\text{Ind}_{P_j(F)}^{G_n(F)}(|\det|^s \tau \otimes \sigma),$$

the convolution $\mathcal{J}(f_{\chi_s} * \Phi_0)$ is supported in the Zariski open orbit, which will be specified below and has the property that

$$\mathcal{Z}_{\nu}(s, f_{\chi_s} * \Phi_0, f_{\mu}, \psi_{\ell, \kappa}) = Q(\chi_s, \mu) \cdot \mathcal{Z}_{\nu}(s, f_{\chi_s}, f_{\mu}, \psi_{\ell, \kappa}).$$

Since $\mathcal{J}(f_{\chi_s} * \Phi_0)$ is supported in the Zariski open orbit, the local zeta integral $\mathcal{Z}_{\nu}(s, f_{\chi_s} * \Phi_0, f_{\mu}, \psi_{\ell, \kappa})$ is entire in s and hence is expected to be $P(\chi_s, \mu)$ essentially.

Let $\mathcal{H}(H_{n-\ell}, K_H)$ be the spherical Hecke algebra with convolution \circ of all K_H -bi-invariant (smooth) functions with compact supports on $H_{n-\ell}$. Let X_i for all $1 \leq i \leq \tilde{m}_H$ be generators of the Hecke algebra $\mathcal{H}(H_{n-\ell}, K_H)$. By the Satake isomorphism, if ν is inert over E and E is the unramified quadratic field extension of F , the Hecke algebra can be realized as follows:

$$\mathcal{H}(H_{n-\ell}, K_H) \simeq \mathbb{C} [X_1, X_1^{-1}, \dots, X_{\tilde{m}_H}, X_{\tilde{m}_H}^{-1}]^{W(H_{n-\ell})};$$

and if ν is split over E , the Hecke algebra can be realized as follows:

$$\mathcal{H}(H_{n-\ell}, K_H) \simeq \mathbb{C} [X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_{m-2\ell-1}^{\pm 1}]^{S_{m-2\ell-1}}.$$

Here $S_{m-2\ell-1}$ is the symmetric group on the sets $\{X_1, \dots, X_{m-2\ell-1}\}$ and $\{X_1^{-1}, \dots, X_{m-2\ell-1}^{-1}\}$.

Define an extended Hecke algebra as in [GPSR97]:

$$\mathcal{A}_{H_{n-\ell}} := \mathbb{C}[X, X^{-1}] \otimes \mathcal{H}(H_{n-\ell}, K_H).$$

Let $\Pi(\chi_s)$ be the unramified representation of $G_n(F)$ as defined in §4.2. We consider the subspace of all K_H -invariant vectors

$$J_{\psi_{\ell,\kappa}}^*(\chi_s) := (J_{\psi_{\ell,\kappa}}(\chi_s))^{K_H}$$

of the twisted Jacquet module $J_{\psi_{\ell,\kappa}}(\chi_s) := J_{\psi_{\ell,\kappa}}(\Pi(\chi_s))$. Although it is naturally a module of the Hecke algebra $\mathcal{H}(H_{n-\ell}, K_H)$, we may extend it to be a module of the extended Hecke algebra $\mathcal{A}_{H_{n-\ell}}$ as follows: for $\phi \in J_{\psi_{\ell,\kappa}}^*(\chi_s)$ and $X \otimes \Phi \in \mathcal{A}_{H_{n-\ell}}$,

$$\phi * (X \otimes \Phi) = q_E^{-s}(\phi \circ \Phi),$$

where $\phi \circ \Phi$ is the left action on ϕ via convolution. As in [GPSR97], define the *support ideal* as follows:

$$\mathcal{I}_{supp}(\chi_s) = \left\{ \Phi \in \mathcal{A}_{H_{n-\ell}} \mid J_{\psi_{\ell,\kappa}}^*(\chi_s) * \Phi \subseteq \Lambda \right\},$$

where Λ is the smooth representation of $H_{n-\ell}(F)$ consisting of functions in $\Pi(\chi_s)$ supported in the open double cosets $P_j \epsilon_{0,1} \eta R_{\ell,w_0}$. More precisely, by Proposition 4.2 and 4.3, the smooth representation Λ can be realized via the following isomorphisms:

$$\Lambda \cong \text{ind}_{P'_{1,\ell}(F)}^{H_{n-\ell}(F)} (|\det|_E^{-\frac{\ell}{2}+s} \tau_{(\ell)} \otimes \sigma^{w_b^\ell})$$

if ν is inert over E ; and

$$\Lambda \cong \text{ind}_{\text{GL}_{m-2j}(F)}^{\text{GL}_{m-2j+1}(F)}(\sigma)$$

if ν is split over E . Here we use the assumption that $j = \ell + 1$.

First we consider the case when $\ell = 0$, which implies that $j = \ell + 1 = 1$. In the case, the twisted Jacquet functor is just the restriction to the subgroup $H_n(F)$ of $G_n(F)$. By restricting to the subgroup $H_n(F)$, the induced representation

$$\Pi = \text{Ind}_{P_1}^{G_n} (|\cdot|_E^s \chi \otimes \sigma)$$

decomposes via an exact sequence of $H_n(F)$ -modules, according to Proposition 4.2.

If ν is inert over E , the case is similar to [GPSR97] and we have

$$0 \rightarrow \text{Ind}_{G_{n-1}}^{H_n}(\sigma) \rightarrow J_{\psi_{0,\kappa}}(\Pi) \rightarrow \text{Ind}_{P'_1}^{H_n} (|t|_E^{\frac{1}{2}+s} \chi \otimes J_{\psi'_{0,\kappa}}(\sigma)) \rightarrow 0.$$

If ν is split over E , more explanation is needed. The double coset decomposition

$$P_{1,m-2,1} \backslash \mathrm{GL}_m / H_n$$

has 6 representatives for $m > 2$, which are denoted by γ_i for $1 \leq i \leq 6$. Let $P_{1,m-2,1}\gamma_1 H_n$ be the open orbit, and $P_{1,m-2,1}\gamma_i H_n$ for $i = 2$ or $i = 3$ be the orbits with the greatest dimension in those orbits except the open orbit. Using Proposition 4.3 repeatedly, we have

$$0 \rightarrow \mathrm{ind}_{\mathrm{GL}_{m-2}}^{\mathrm{GL}_{m-1}}(\sigma) \rightarrow \Omega \rightarrow \Sigma \rightarrow 0,$$

where

$$(4.11) \quad \Omega := \{f \in \Pi \mid \mathrm{supp}(f) \subseteq \cup_{i=1}^3 P_{1,m-2,1}\gamma_i H_n\},$$

and

$$\Sigma := \mathrm{Ind}_{P_{1,m-2}}^{H_n}(| \cdot |^{\frac{1}{2}+s} \Theta \otimes \sigma_{[0]}) \oplus \mathrm{Ind}_{P_{m-2,1}}^{H_n}(\sigma_{[0]} \otimes | \cdot |^{-\frac{1}{2}-s} \Xi^{-1}).$$

Lemma 4.4. *Assume that $\ell = 0$ and $j = \ell + 1 = 1$. The support ideal $\mathcal{I}_{\mathrm{supp}}(\chi_s)$ contains*

$$\Phi_0 = \prod_i (1 - q_E^{-\frac{1}{2}} \chi(\varpi) X X_i) (1 - q_E^{-\frac{1}{2}} \chi(\varpi) X X_i^{-1})$$

if ν is inert over E , and

$$\Phi_0 = \prod_i (1 - q_E^{-\frac{1}{2}} \Theta(\varpi) X X_i) (1 - q_E^{-\frac{1}{2}} \Xi(\varpi) X X_i^{-1})$$

if ν is split over E .

Proof. The proof follows the same argument used in [GPSR97, §2, Lemma 2.1], which uses the Satake Isomorphism for F -quasisplit classical groups and the definition of the support ideal $\mathcal{I}_{\mathrm{supp}}(\chi_s)$. We omit the details here. \square

Next, we deal with the general case with $j = \ell + 1$ for the relation between $H_{n-\ell}$ and $G_{n-\ell}$.

If ν is inert over E , by Proposition 4.2, we have the exact sequence of $H_{n-\ell}(E)$ modules for $j = \ell + 1$,

$$0 \rightarrow \mathrm{ind}_{P'_1}^{H_{n-\ell}} | \cdot |_E^{s-\frac{\ell}{2}} \tau^{(\ell)} \otimes \sigma^{w_q^\ell} \rightarrow \Pi_{\epsilon_{0,1}\eta_{\epsilon,I_{m-2\ell}}} \rightarrow \mathcal{Y} \rightarrow 0$$

where

$$\mathcal{Y} := \mathrm{ind}_{P'_1}^{H_{n-\ell}} | \cdot |_E^{\frac{1-\ell}{2}+s} \tau^{(\ell)} \otimes J_{\psi'_{0,\kappa}}(\sigma^{w_q^\ell}),$$

and $\Pi_{\epsilon_{0,1}\eta_{\epsilon,I_{m-2\ell}}}$ is the smooth representation of $H_{n-\ell}$ consisting of functions in $\pi(\chi_s)$ which are supported in $P_j \epsilon_{0,1}\eta_{\epsilon,I_{m-2\ell}} N_\ell G_{n-\ell}$. Recall that

$\tau^{(\ell)}$ is the ℓ -th Bernstein-Zelevinski derivative of τ , which is a representation of $\mathrm{GL}_1(E)$. Up to semi-simplification,

$$\tau^{(\ell)} = \oplus_{i=1}^j \chi_i \otimes |\cdot|^{-\frac{\ell}{2}},$$

and then $\mathcal{Y} \equiv \oplus_{i=1}^j \mathrm{ind}_{P'_1}^{H_{n-\ell}} |\det|^{-\frac{1}{2}+s}_E \chi_i \otimes J_{\psi'_{0,\kappa}}(\sigma^{w_q})$.

If ν is split, we apply Proposition 4.3 repeatedly and obtain the exact sequence

$$0 \rightarrow \mathrm{ind}_{\mathrm{GL}_{m-2j}}^{\mathrm{GL}_{m-2\ell-1}} \sigma_{(0)} \rightarrow \Omega \rightarrow \mathcal{U} \rightarrow 0,$$

where \mathcal{U} is defined to be the following representation

$$\mathrm{Ind}_{P_{1,m-2j}}^{\mathrm{GL}_{m-2\ell-1}} (|\det|^{-\frac{1-\ell}{2}+s} (\tau_1^{(\ell)} \otimes \sigma) \oplus \mathrm{Ind}_{P_{1,m-2j}}^{\mathrm{GL}_{m-2\ell-1}} (\sigma \otimes |\det|^{-\frac{\ell-1}{2}-s} (\tau_2^*)^{[\ell]})$$

and Ω is defined in (4.11) consisting of functions supported in the first greatest orbits. In this case, we have, up to semi-simplification,

$$\tau_1^{(\ell)} = \oplus_{i=1}^j \Theta_i \otimes |\cdot|^{-\frac{\ell}{2}} \text{ and } (\tau_2^*)^{[\ell]} = \oplus_{i=1}^j \Xi_i^{-1} \otimes |\cdot|^{-\frac{\ell}{2}}.$$

Note that $\Phi \in \mathcal{I}_{\mathrm{supp}}(\chi_s)$ if and only if Φ annihilates all the boundary components of $J_{\psi_{\ell,\kappa}}(\Pi)$, that is, all the summands in Proposition 4.2 and Proposition 4.3 except the space $\mathrm{ind}_{P'_{1,\ell}}^{H_{n-\ell}} (|\det|^{-\frac{\ell}{2}+s}_E \tau_{(\ell)} \otimes \sigma^{w_b})$ and the space $\mathrm{ind}_{\mathrm{GL}_{m-2j}}^{\mathrm{GL}_{m-2\ell-1}} \sigma$, respectively. It is sufficient to annihilate the quotients in $\Pi_{\epsilon_{0,1}}$ and Ω .

In order to annihilate K_H -fixed vectors in the space

$$\oplus_{i=1}^j \mathrm{ind}_{P'_1}^{H_{n-\ell}} (|\det|^{-\frac{1}{2}+s}_E \chi_i \otimes J_{\psi'_{0,\kappa}}(\sigma^{w_b}))$$

if ν is inert, and in the space

$$\oplus_{i=1}^j \mathrm{Ind}_{P_{1,m-2j}}^{\mathrm{GL}_{m-2\ell-1}} (|\cdot|^{-\frac{1}{2}+s} \Theta_i \oplus |\cdot|^{-\frac{1}{2}-s} \Xi_i^{-1}) \otimes \sigma$$

(up to isomorphism) if ν is split, as in Lemma 4.4, we may take the following specific element in $\mathcal{A}_{H_{n-\ell}}$,

$$(4.12) \quad \Phi_0 = \begin{cases} \prod_{i=1}^j \prod_{i'=1}^{\tilde{m}_H} (1 - q_E^{-\frac{1}{2}} \chi_i X X_{i'}) (1 - q_E^{-\frac{1}{2}} \chi_i X X_{i'}^{-1}) & \text{if } \nu \text{ is inert,} \\ \prod_{i=1}^j \prod_{i'=1}^{m-2\ell-1} (1 - q_E^{-\frac{1}{2}} \Theta_i X X_{i'}) (1 - q_E^{-\frac{1}{2}} \Xi_i X X_{i'}^{-1}) & \text{if } \nu \text{ is split,} \end{cases}$$

which is an element of the support ideal $\mathcal{I}_{\mathrm{supp}}(\chi_s)$. In addition, all the other boundary components of the Jacquet module $J_{\psi_{\ell,\kappa}}(\chi_s)$ are of form

$$\mathrm{ind}_{P'_\beta}^{H_{n-\ell}} (|\det|^{-\frac{1-t}{2}+s}_E \tau^{(t)} \otimes \sigma')$$

if ν is inert, and of form

$$\mathrm{ind}_{P_{\beta,m-2\ell-1-\beta}}^{\mathrm{GL}_{m-2\ell-1}} (|\det|^{-\frac{1-t}{2}+s}_E \Xi_\tau \otimes \sigma')$$

or

$$\mathrm{ind}_{P_{m-2\ell-1-\beta,\beta}}^{\mathrm{GL}_{m-2\ell-1}}(\sigma' \otimes |\det|^{-\frac{1-t}{2}-s}\Theta_\tau)$$

if ν is split, where σ' is a suitable representation independent of s , more details of which can be found in Proposition 4.2 and Proposition 4.3. It is easy to check that Φ_0 also annihilates K_H -fixed vectors in those boundary components.

Proposition 4.5. *With $\Phi_0 \in \mathcal{I}_{\mathrm{supp}}(\chi_s)$ as chosen above, the following identity holds:*

$$(4.13) \quad \mathcal{Z}_\nu(s, f_\chi * \Phi_0, f_\mu, \psi_{\ell,\kappa}) = Q(\chi_s, \mu) \cdot \mathcal{Z}_\nu(s, f_\chi, f_\mu, \psi_{\ell,\kappa})$$

where $Q(\chi_s, \mu)$ is defined to be

$$\prod_{i=1}^j \prod_{i'=1}^{\tilde{m}_H} (1 - q_E^{-\frac{1}{2}-s} \chi_i \mu_{i'}) (1 - q_E^{-\frac{1}{2}-s} \chi_i \mu_{i'}^{-1})$$

if ν is inert, and to be

$$\prod_{i=1}^j \prod_{i'=1}^{m-2\ell-1} (1 - q_E^{-\frac{1}{2}-s} \Theta_i \mu_{i'}) (1 - q_E^{-\frac{1}{2}-s} \Xi_i \mu_{i'}^{-1})$$

if ν is split. Moreover, $\mathcal{Z}_\nu(s, f_{\chi_s} * \Phi_0, f_\mu, \psi_{\ell,\kappa})$ is a polynomial function of parameters χ_τ and in q_E^{-s} .

Proof. The proof is similar to the proof of Theorem 5.1 in [GPSR97]. In fact, $\mathcal{J}(f_{\chi_s} * \Phi_0)(h)$, as defined in Section 4.1, belongs to the space Λ , which is independent of the choice of σ . Also $\mathcal{J}(f_{\chi_s} * \Phi_0)(h)$ is analytic in s because of the support of $\mathcal{J}(f_{\chi_s} * \Phi_0)$. The local zeta integral is equal to the pairing the function $\mathcal{J}(f_{\chi_s} * \Phi_0)(h)$ with a Bessel function as in (4.3), and is absolutely convergent for all s . Hence the zeta function $\mathcal{Z}_\nu(s, f_{\chi_s} * \Phi_0, f_\mu, \psi_{\ell,\kappa})$ is a polynomial function of q_E^s and q_E^{-s} for all choice of π and all s . \square

Remark that the proof of this proposition only uses the genericity of τ , which is true because τ is the unramified local component of the corresponding irreducible automorphic representation of $\mathrm{GL}_j(\mathbb{A}_E)$ as given in (2.13). Hence it holds for all choices χ_σ and μ , and therefore, for all irreducible unramified representations σ and π .

Following the definition of the unramified local tensor product L -functions as in (4.5) and (4.6), and Proposition 4.5, one must have the following identity:

$$Q(\chi_s, \mu) = \begin{cases} L^{-1}(\frac{1}{2} + s, \tau \times \pi) d(\chi_\tau, s) & \text{if } \nu \text{ is inert,} \\ L^{-1}(\frac{1}{2} + s, \tau \times \pi) & \text{if } \nu \text{ is split.} \end{cases}$$

where

$$d(\chi_\tau, s) = \begin{cases} \prod_{i=1}^j (1 - q_E^{-\frac{1}{2}-s} \chi_i)^{-1} & \text{if } m \text{ is even and } \nu \text{ is inert;} \\ 1 & \text{otherwise.} \end{cases}$$

Note that $d(\chi_i, s) = (1 - q_E^{-\frac{1}{2}-s} \chi_i)^{-1}$. Thus, based on the calculation of $Q(\chi_s, \mu)$, we have a unique choice of $P(\chi_s, \mu)$.

4.5. Calculation of $P(\chi_s, \mu)$. In this section, we will first calculate the numerator $P(\chi_s, \mu)$ when Π and π are *generic spherical*, and then extend the results to general case by *Density Principle* in Appendix IV to [GPSR97, Section 5].

Choose suitable functions $\varphi_1 \in \mathcal{S}(G_n)$ and $\varphi_2 \in \mathcal{S}(H_{n-\ell})$ such that

$$\begin{aligned} f_{\chi_s}(g) &= \int_B \varphi_1(bg) \chi_s^{-1} \delta_B^{\frac{1}{2}}(b) \, d_l b, \\ f_\mu(g) &= \int_{B_H} \varphi_2(bg) \mu^{-1} \delta_{B_H}^{\frac{1}{2}}(b) \, d_l b, \end{aligned}$$

where $d_l b$ is the left invariant Haar measure on Borel subgroups. Then we define a linear functional in the hom space (4.2),

$$T(f_{\chi_s}, f_\mu) := \int_{R_{\ell, \kappa}} f_{\chi_s}(\epsilon_{0,1} \eta n m) f_\mu(m) \psi_{\ell, \kappa}(n) \, dn \, dm.$$

Remark that properties of T are studied in [K08] when ν is inert.

Lemma 4.6.

$$\mathcal{Z}_\nu(s, f_{\chi_s}, f_\mu, \psi_{\ell, \kappa}) = T(f_{\chi_s}, f_\mu).$$

Proof. For all unramified places, the proof is similar to the orthogonal case as Theorem (A) in Appendix I to [GPSR97, Chapter 5]. \square

Case $\ell = 0$: First of all, we consider the case $\ell = 0$, and hence $j = 1$. The Bessel period is also studied by Gan, Gross, and Prasad in [GGP12]. Referring to [Har12, Proposition 2.5], we have the following inductive formula

$$T(f_{\chi_1 \otimes \chi_\sigma}, f_\mu) = \frac{L(\frac{1}{2}, \chi_1 \times \pi)}{L(1, \chi_1 \times \sigma) L(1, \xi_{E/F}^m \otimes \chi_1)} T(f_\mu, f_{\chi_\sigma})$$

if ν is inert, and

$$T(f_{\chi_1 \otimes \chi_\sigma}, f_\mu) = \frac{L(\frac{1}{2}, \Theta_1 \times \pi) L(\frac{1}{2}, \Xi_1 \times \tilde{\pi})}{L(1, \Theta_1 \times \tilde{\sigma}) L(1, \Xi_1 \times \sigma) L(1, \Theta_1 \Xi_1)}$$

if ν is inert, for any quasi-character χ_1 . Correspondingly, one has

$$Q(\chi_1 \otimes \chi_\sigma, \mu) = \begin{cases} [L(\frac{1}{2}, \chi_1 \times \pi) d(\chi_1)]^{-1} & \text{if } \nu \text{ is inert,} \\ [L(\frac{1}{2}, \Theta_1 \times \pi) L(\frac{1}{2}, \Xi_1 \times \tilde{\pi})]^{-1} & \text{if } \nu \text{ is split.} \end{cases}$$

which is the same as the result of Proposition 4.5. Hence one has

$$(4.14) \quad P(\chi_1 \otimes \chi_\sigma, \mu) = \frac{d(\chi_1)}{L(1, \chi_1 \times \sigma) L(1, \chi_1 \otimes \xi_{E/F}^m)} T(\mu, \chi_\sigma).$$

Note that $P(\chi_1 \otimes \chi_\sigma, \mu)$ is a polynomial function of the parameter χ_1 , and $L(1, \chi_1 \otimes \xi_{E/F}^m) = L(1, \Theta_1 \otimes \Xi_1)$. A comment with the notation χ_1 is in order. The above discussion holds for all quasi-characters χ_1 and hence the variable s is carried by this character χ_1 here.

General Case $\ell > 0$: In the discussion below, we also assume that χ is a general quasi-character, i.e. we take χ to be χ_s here, since the proof works for any quasi-character χ . Hence in the discussion, there will be no variable s . However, the variable s will be put back to the final formula.

Let ω be an element of Weyl group $W(H_{n-\ell})$ and I_ω be the intertwining operator mapping $\Pi(\chi)$ to $\Pi(\omega\chi)$. By the uniqueness of Bessel model, we have a local gamma factor $\gamma_\omega(\chi, \gamma)$ defined by

$$T(I_\omega(f_\chi), f_\mu) = \gamma_\omega(\chi, \mu) T(f_\chi, f_\mu).$$

Note that the definition of γ_ω is independent with non-trivial choice of T . In order to calculate the general case $\ell > 0$. We need to calculate the local gamma factor γ_ω .

When ν is inert, let $\{\beta_i \mid 1 \leq i \leq \tilde{m}\}$ be a set of simple roots of G_n . Then the sets $\{\beta_i \mid 1 \leq i \leq \ell\}$ and $\{\beta_i \mid \ell + 2 \leq i \leq \tilde{m}\}$ are also sets of simple roots of $\mathrm{GL}_{\ell+1}(E)$ and $H(W_{\ell+1})$ respectively.

When ν is split, let $\{\beta_i \mid 1 \leq i \leq m-1\}$ be a set of simple roots of GL_m . Recall that $P_{\ell+1, m-2\ell-2, \ell+1}$ is a standard parabolic subgroup of GL_m with the Levi subgroup $\mathrm{GL}_{\ell+1} \times \mathrm{GL}_{m-2\ell-2} \times \mathrm{GL}_{\ell+1}$. Then the set $\{\beta_i \mid 1 \leq i \leq \ell\}$ and $\{\beta_i \mid m-\ell \leq i \leq m-1\}$ are sets of simple roots of the general linear groups of the Levi subgroup, and $\{\beta_i \mid \ell+2 \leq i \leq m-\ell-2\}$ is the set of simple roots of the subgroup $\mathrm{GL}_{m-2\ell-2}$ of the Levi subgroup. Let ω_i be the simple reflection corresponding to the simple root β_i .

Lemma 4.7. *If ν is inert, then*

$$\gamma_{\omega_i}(\chi, \mu) = \begin{cases} \frac{1 - \chi_{i+1} \chi_i^{-1} q_E^{-1}}{1 - \chi_i \chi_{i+1}^{-1}} & \text{if } 1 \leq i \leq \ell, \\ \gamma_{\omega_i}(\chi_{\ell+1} \otimes \chi_\sigma, \mu) & \text{if } \ell+1 \leq i \leq \tilde{m}. \end{cases}$$

If ν is split, then the gamma factor $\gamma_{\omega_i}(\chi, \mu)$ is equal to

$$\begin{cases} \frac{1 - \chi_{i+1}\chi_i^{-1}q_E^{-1}}{1 - \chi_i\chi_{i+1}^{-1}} & \text{if } 1 \leq i \leq \ell \text{ or } m - \ell \leq i \leq m, \\ \gamma_{\omega_i}(\chi_{\ell+1} \otimes \chi_\sigma \otimes \chi_{m-\ell}, \mu) & \text{if } \ell + 1 \leq i \leq m - \ell - 1. \end{cases}$$

Proof. Khoury proved the inert case in [K08, Proposition 11.1]. For the split case, the proof is given in [Z12]. \square

Now, we normalize the numerator $P(\chi, \mu)$ by

$$P^*(\chi, \mu) = \frac{\zeta(\chi, 1)T(\mu, \chi_\sigma)}{P(\chi, \mu)}.$$

Note that $T(\mu, \chi_\sigma)$ is the pairing for $\pi(\mu)$ and σ .

Following [CS80] and [Sh10, Section 3.5], the functions $\zeta(\chi, t)$ can be defined as follows. When ν is inert, if m is even, $\zeta(\chi, t)$ is defined by

$$\prod_{1 \leq i_1 < i_2 \leq \tilde{m}} (1 - \chi_{i_1}\chi_{i_2}^{-1}q^{-t})(1 - \chi_{i_1}\chi_{i_2}q^{-t}) \cdot \prod_{1 \leq i \leq \tilde{m}} (1 - \chi_i q_F^{-t});$$

and if m is odd, $\zeta(\chi, t)$ is defined by

$$\prod_{1 \leq i_1 < i_2 \leq \tilde{m}} (1 - \chi_{i_1}\chi_{i_2}^{-1}q^{-t})(1 - \chi_{i_1}\chi_{i_2}q^{-t}) \cdot \prod_{1 \leq i \leq \tilde{m}} (1 + \chi_i q_F^{-t})(1 - \chi_i q^{-t}).$$

When ν is split, $\zeta(\chi, t) = \prod_{1 \leq i_1 < i_2 \leq m} (1 - \chi_{i_1}\chi_{i_2}^{-1}q^{-t})$. Note that $q = q_E$ in the above formulas, in order to simplify the notation. In addition, if $\tilde{m} = 1$, $\zeta(\chi, t) = 1$ for all cases. Remark that $\zeta(\chi, t)$ is the zeta polynomial function associated to G_n as in [GPSR97, Page 157].

For the case $\ell = 0$, according to (4.14), we have

$$(4.15) \quad P^*(\chi_1 \otimes \chi_\sigma, \mu) = \frac{\zeta(\chi_\sigma, 1)}{d(\chi_1)},$$

where $\zeta(\chi_\sigma, 1)$ is the zeta polynomial function associated with H_n , as in [GPSR97, Page 157].

Corollary 4.8. *If $1 \leq i \leq \ell$, or $m - \ell \leq i \leq m - 1$ when ν is split, then*

$$P^*(\omega_i \chi, \mu) = P^*(\chi, \mu).$$

If $i = \ell + 1$ when ν is inert, or $i = \ell + 1$ or $m - \ell - 1$ when ν is split, then

$$\frac{P^*(\chi, \mu)}{P^*(\omega_i \chi, \mu)} = \frac{\zeta(\chi_\sigma, 1)d(\chi_i)}{\zeta(\chi_{\sigma'}, 1)d(\chi_{i+1})}.$$

where $\chi_{\sigma'} = \chi_{\ell+1} \otimes \chi_{\ell+3} \otimes \cdots \otimes \chi_{\tilde{m}}$ when $i = \ell_1$ and ν is inert, and $\chi_{\sigma'} = \chi_{\ell+1} \otimes \chi_{\ell+3} \otimes \cdots \otimes \chi_{m-\ell-1}$ when $i = \ell$ and ν is split, and $\chi_{\sigma'} = \chi_{\ell+2} \otimes \cdots \otimes \chi_{m-\ell-2} \otimes \chi_{m-\ell}$.

If $\ell + 1 < i \leq \tilde{m}$ when ν is inert or $\ell + 2 \leq i \leq m - \ell - 2$ when ν is split, then

$$\frac{P^*(\chi, \mu)}{P^*(\omega_i \chi, \mu)} = \frac{\zeta(\chi_\sigma, 1)}{\zeta((\omega \chi)_\sigma, 1)}.$$

Proof. The proof is a straightforward calculation by Lemma 4.7. \square

By Corollary 4.8,

$$\frac{P^*(\chi, \mu) d(\chi_\tau)}{\zeta(\chi_\sigma, 1)}$$

is invariant under the action of the Weyl group $W(G_n)$ on χ . In the rest of this section, we will show that the quotient above is equal to one, i.e.

$$(4.16) \quad P^*(\chi, \mu) = \frac{\zeta(\chi_\sigma, 1)}{d(\chi_\tau)}.$$

Let $T_0(\chi, \mu) = T(f_\chi^0, f_\mu)$, where

$$f_\chi^0(g) = \int_B 1_{B(\mathfrak{o})\omega_{G_n}B(\mathfrak{o})}(bg)\chi^{-1}\delta_B^{\frac{1}{2}}(b) \, d_l b$$

ω_{G_n} is the longest Weyl element in G_n , and $1_{B(\mathfrak{o})\omega_{G_n}B(\mathfrak{o})}$ is the characteristic function over $B(\mathfrak{o})\omega_{G_n}B(\mathfrak{o})$ and also is an Iwahori-fixed function.

Lemma 4.9.

$$T_0(\chi, \mu) = T_0(\chi|_{H_{n-\ell}}, \mu).$$

Proof. The proof is similar to Proposition 8.1 in [GPSR97]. \square

By Appendix to §6 in [GPSR97], we have the following expansion,

$$T(\chi, \mu) = \sum_{\omega \in W(G_n)} \frac{\gamma_\omega(\omega^{-1}\chi, \mu)}{c_\omega(\omega^{-1}\chi)} c_{\omega_{G_n}}(\omega^{-1}\chi) T_0(\omega^{-1}\chi, \mu),$$

where $c_\omega(\omega^{-1}\chi)$ is the Harish-Chandra c -function of the intertwining operator associated to the Weyl group element ω . In this formula, by replacing $\gamma_\omega(\omega^{-1}\chi, \mu)$ by the following expression:

$$\gamma_\omega(\omega^{-1}\chi, \mu) = \frac{T(\chi, \mu) c_\omega(\omega^{-1}\chi)}{T(\omega^{-1}\chi, \mu)},$$

canceling both sides the factor $T(\chi, \mu)$, and replacing $T(\omega^{-1}\chi, \mu)$ by

$$T(\omega^{-1}\chi, \mu) = \frac{P(\omega^{-1}\chi, \mu)}{Q(\omega^{-1}\chi, \mu)},$$

we obtain the following expression:

$$\begin{aligned} 1 &= \sum_{\omega \in W(G_n)} \frac{Q(\omega^{-1}\chi, \mu)}{P(\omega^{-1}\chi, \mu)} c_{\omega_{G_n}}(\omega^{-1}\chi) T_0(\omega^{-1}\chi, \mu) \\ &= \sum_{\omega \in W(G_n)} \frac{c_{\omega_{G_n}}(\omega^{-1}\chi)}{\zeta(\omega^{-1}\chi, 1)} Q(\omega^{-1}\chi, \mu) P^*(\omega^{-1}\chi, \mu) \frac{T_0(\omega^{-1}\chi, \mu)}{T(\mu, (\omega^{-1}\chi)_\sigma)}. \end{aligned}$$

Define

$$\Delta(\chi) = q^{\langle \varrho, \chi \rangle} \zeta(\chi, 0) = \prod_{i=1}^{\tilde{m}} \chi_i^{-(\frac{m+1}{2}-i)} \zeta(\chi, 0),$$

where ϱ is the half of the sum of all positive roots. Then it follows that $\Delta(\omega\chi) = \text{sgn}(\omega)\Delta(\chi)$. Note that $c_{\omega_{G_n}}(\chi) = \zeta(\chi, 1)\zeta^{-1}(\chi, 0)$. It follows that $\Delta(\chi)$ can be expressed as follows:

$$\begin{aligned} (4.17) \quad & \sum_{\omega \in W(G_n)} \text{sgn}(\omega) q^{\langle \varrho, \omega^{-1}\chi \rangle} Q(\omega^{-1}\chi, \mu) P^*(\omega^{-1}\chi, \mu) \frac{T_0(\omega^{-1}\chi, \mu)}{T(\mu, (\omega^{-1}\chi)_\sigma)} \\ &= \frac{P^*(\chi, \mu) d(\chi_\tau)}{\zeta(\chi_\sigma, 1)} \sum_{\omega \in W(G_n)} \text{sgn}(\omega) q^{\langle \varrho, \omega\chi \rangle} Q(\omega\chi, \mu) \frac{T_0(\omega\chi, \mu) \zeta((\omega\chi)_\sigma, 1)}{T(\mu, (\omega\chi)_\sigma) d((\omega\chi)_\tau)}. \end{aligned}$$

In order to prove Equation (4.16), it is sufficient to show the following Lemma, which is similar to the orthogonal case ([GPSR97, Lemma 6.3]).

Lemma 4.10.

$$(4.18) \quad \Delta(\chi) = \sum_{\omega \in W(G_n)} \text{sgn}(\omega) q^{\langle \varrho, \omega\chi \rangle} Q(\omega\chi, \mu) \frac{T_0((\omega\chi)|_{H_{n-\ell}}, \mu) \zeta((\omega\chi)_\sigma, 1)}{T(\mu, (\omega\chi)_\sigma) d((\omega\chi)_\tau)}.$$

Proof. We only give a proof for the inert case. For the split case, the proof is similar and we omit details here.

First, by Equation (4.15), this identity holds for $\ell = 0$.

Next, we consider the general cases $\ell > 0$. Since the terms $\zeta(\chi_\sigma, 1)$, $Q(\chi, \mu)$, $d(\chi_\tau)$, $T_0(\chi|_{H_{n-\ell}}, \mu)$ and $T(\mu, (\omega\chi)_\sigma)$ are invariant under the action of the Weyl group $W(\text{GL}_{\ell+1})$, we have the right hand side of the identity,

$$RHS = \sum_{\omega \in W(GL_\ell) \times W(G_{n-\ell}) \setminus W(G_n)} \Sigma_{\omega_1}(\omega) \cdot \Sigma_{\omega_2}(\omega) \cdot q^{\langle \varrho_{U_\ell}, \omega\chi \rangle} \text{sgn}(\omega).$$

where

$$\Sigma_{\omega_1}(\omega) := \sum_{\omega_1 \in W(GL_\ell)} \text{sgn}(\omega_1) q^{\langle \varrho_{GL_\ell}, \omega_1 \omega\chi \rangle},$$

and

$$\begin{aligned} \Sigma_{\omega_2}(\omega) &:= \sum_{\omega_2 \in W(G_{n-\ell})} \text{sgn}(\omega_2) q^{\langle \varrho_{G_{n-\ell}}, \omega_2 \omega \chi \rangle} Q(\omega_2 \omega \chi, \mu) \\ &\quad \cdot \frac{T_0((\omega_2 \omega \chi)|_{H_{n-\ell}}, \mu) \zeta((\omega_2 \omega \chi)_\sigma, 1)}{T(\mu, (\omega_2 \omega \chi)_\sigma) d((\omega_2 \omega \chi)_\tau)}. \end{aligned}$$

Decompose as $Q(\chi, \mu) = Q_1(\chi, \mu) Q(\chi_{\ell+1} \otimes \chi_\sigma, \mu)$, where

$$Q_1(\chi, \mu) = \prod_{i=1}^{\ell} \prod_{i'=1}^{\tilde{m}_L} (1 - q^{-\frac{1}{2}} \chi_i \mu_{i'}^{-1}) (1 - q^{-\frac{1}{2}} \chi_i \mu_{i'}^{-1})$$

and

$$Q(\chi_{\ell+1} \otimes \chi_\sigma, \mu) = \prod_{i'=1}^{\tilde{m}_L} (1 - q^{-\frac{1}{2}} \chi_{\ell+1} \mu_{i'}^{-1}) (1 - q^{-\frac{1}{2}} \chi_{\ell+1} \mu_{i'}^{-1}).$$

Thus, $Q(\omega_2 \chi, \mu) = Q_1(\chi, \mu) Q(\omega_2(\chi_{\ell+1} \otimes \chi_\sigma), \mu)$ for $\omega_2 \in W(G_{n-\ell})$.

Define $\omega \chi = \chi^{(1)} \otimes \chi^{(2)}$, where $\chi^{(1)} = \omega \chi|_{\text{GL}_\ell}$ and $\chi^{(2)} = \omega \chi|_{G_{n-\ell}}$. Note that

$$\begin{aligned} \langle \varrho_{H_{n-\ell}}, \omega_2 \omega \chi \rangle &= \langle \varrho_{H_{n-\ell}}, \omega_2 \chi^{(2)} \rangle, \\ \zeta((\omega_2 \omega \chi)_\sigma, 1) &= \zeta((\omega_2 \chi^{(2)})_\sigma, 1), \end{aligned}$$

and

$$\begin{aligned} d((\omega_2 \omega \chi)_\tau) &= d((\omega_2 \chi^{(2)})_{\ell+1}) \prod_{i=1}^{\ell} (1 - q^{-1} \chi_i(\varpi_E)) \\ &= d((\omega_2 \chi^{(2)})_{\ell+1}) d((\omega \chi)_\tau) d^{-1}((\omega \chi)_{\ell+1}). \end{aligned}$$

Consider the summation

$$\begin{aligned} \Sigma_{\omega_2}(\omega) &= Q_1(\omega \chi, \mu) \frac{d((\omega \chi)_{\ell+1})}{d((\omega \chi)_\tau)} \\ &\quad \cdot \sum_{\omega_2 \in W(G_{n-\ell})} \text{sgn}(\omega_2) q^{\langle \varrho_{G_{n-\ell}}, \omega_2 \chi^{(2)} \rangle} Q(\omega_2 \chi^{(2)}, \mu) \\ &\quad \cdot \frac{T_0(\omega_2 \chi^{(2)}, \mu) \zeta((\omega_2 \chi^{(2)})_\sigma, 1)}{T(\mu, \omega_2 \chi^{(2)}) d((\omega_2 \chi^{(2)})_{\ell+1})} \\ &= Q_1(\omega \chi, \mu) \frac{d((\omega \chi)_{\ell+1})}{d((\omega \chi)_\tau)} \cdot \Delta_{H_{n-\ell}}(\chi^{(2)}). \end{aligned}$$

The last identity holds by the case $\ell = 0$. Note that $\chi^{(2)} = \omega \chi|_{G_{n-\ell}}$

Now, by replacing $\Sigma_{\omega_2}(\omega)$ by the expression above, the right hand side of (4.18) reduces to

$$\begin{aligned} RHS = & \sum_{\omega \in W(\mathrm{GL}_\ell) \times W(G_{n-\ell}) \setminus W(G_n)} \Sigma_{\omega_1}(\omega) \\ & \cdot Q_1(\omega\chi, \mu) \frac{d((\omega\chi)_{\ell+1})}{d((\omega\chi)_\tau)} \cdot \Delta_{H_{n-\ell}}(\chi^{(2)}) \mathrm{sgn}(\omega) q^{\langle \varrho_{U_\ell}, \omega\chi \rangle}. \end{aligned}$$

By using the definition of $\Sigma_{\omega_1}(\omega)$ and the definition of $\Delta_{H_{n-\ell}}(\chi^{(2)})$, and then by collapsing the three summations \sum_ω , \sum_{ω_1} and \sum_{ω_2} , we obtain that

$$RHS = \sum_{\omega \in W(G_n)} \mathrm{sgn}(\omega) q^{\langle \varrho, \omega\chi \rangle} Q_1(\omega\chi, \mu) \frac{d((\omega\chi)_{\ell+1})}{d((\omega\chi)_\tau)}.$$

Recall that

$$Q_1(\chi, \mu) \frac{d((\chi)_{\ell+1})}{d((\chi)_\tau)} = \prod_{i=1}^{\ell} \prod_{i'=1}^{\tilde{m}_H} (1 - q^{-\frac{1}{2}} \chi_i \mu_{i'}^{-1}) (1 - q^{-\frac{1}{2}} \chi_i \mu_{i'}^{-1}).$$

Then

$$\begin{aligned} RHS = & \sum_{\omega \in W(G_n)} \mathrm{sgn}(\omega) q^{\langle \varrho, \omega\chi \rangle} \\ & + \sum_{\vec{n}} c_{\vec{n}} \sum_{\omega \in W(G_n)} \mathrm{sgn}(\omega) q^{\langle \varrho, \omega\chi \rangle} \prod_{i=1}^{\ell} \chi_i^{n_i}, \end{aligned}$$

where $\vec{n} = (n_1, n_2, \dots, n_\ell)$ with $n_i \in \{0, 1, 2\}$ such that at least one n_i is nonzero, and $c_{\vec{n}}$ is the coefficient depending only on μ . Also note that

$$q^{\langle \varrho, \chi \rangle} \prod_{i=1}^{\ell} \chi_i^{n_i} = \prod_{i=1}^{\tilde{m}} \chi_i^{-(\frac{m+1}{2} - i - n_i)},$$

where $n_i = 0$ if $i > \ell$. Thus, it is sufficient to show that

$$\sum_{\omega \in W(G_n)} \mathrm{sgn}(\omega) q^{\langle \varrho, \omega\chi \rangle} \prod_{i=1}^{\ell} (\omega\chi)_i^{n_i} = 0.$$

Since $\sum_{i=1}^{\ell} n_i \neq 0$, $\ell > 0$ and $\tilde{m} - \ell - 1 \geq 1$, there exist at least two distinct integers i and i' with $i < i'$ such that $\frac{m+1}{2} - i - n_i = \frac{m+1}{2} - i' - n_{i'}$. Let i_0 be the maximal integer such that $i_0 + n_{i_0} = i'_0 + n_{i'_0}$. Consider the Weyl group $W(G_n)$ as the subgroup of the permutation group on χ_i and χ_i^{-1} for $1 \leq i \leq \tilde{m}$. Then, define a Weyl element ω' by the following rules: ω' permutes χ_{i_0} and $\chi_{i'_0}$, and fixes χ_i for the rest i . Hence, $\mathrm{sgn}(\omega') = -1$ and ω' fixes $\prod_{i=1}^{\tilde{m}} \chi_i^{-(\frac{m+1}{2} - i - n_i)}$. Let $W(G_n)_{\vec{n}}$ be

the stabilizer of $W(G_n)$ acting on $q^{\langle \varrho, \omega \chi \rangle} \prod_{i=1}^{\ell} \chi_i^{n_i}$. By $\text{sgn}(\omega') = -1$ and $\omega' \in W(G_n)_{\tilde{n}}$, we have the restriction of sgn on $W(G_n)_{\tilde{n}}$ is not trivial.

Therefore,

$$\begin{aligned} & \sum_{\omega \in W(G_n)} \text{sgn}(\omega) q^{\langle \varrho, \omega \chi \rangle} \prod_i^{\ell} (\omega \chi)_i^{n_i} \\ &= \sum_{\omega \in W(G_n)} q^{\langle \varrho, \omega \chi \rangle} \prod_i^{\ell} (\omega \chi)_i^{n_i} \sum_{\omega' \in W(G_n)_{\tilde{n}}} \text{sgn}(\omega \omega') \\ &= 0. \end{aligned}$$

□

Comparing (4.17) and (4.18), we can get the identity (4.16). Hence, after replacing back χ_s for χ , we obtain the following formulas

$$(4.19) \quad P(\chi_s, \mu) = \frac{d((\chi_s)_{\tau}) \zeta((\chi_s)_{\tau}, 1)}{L(s+1, \tau \times \sigma) L(2s+1, \tau, \text{Asai} \otimes \xi_{E/F}^m)} T(\mu, \chi_{\sigma})$$

if ν is inert, and

$$(4.20) \quad P(\chi_s, \mu) = \frac{\zeta((\chi_s)_{\tau_1}, 1) \zeta((\chi_s)_{\tau_2}, 1)}{L(s+1, \tau_1 \times \tilde{\sigma}) L(s+1, \tau_2 \times \sigma) L(2s+1, \tau_1 \times \tau_2)}$$

if ν is split. Note that $(\chi_s)_{\tau}$ denotes the quasi-character which is the restriction of the quasi-character χ_s to the τ -part of the torus.

It is important to point out that from the beginning of this section up to this point, we assume that $\Pi(\chi_s)$ and $\pi(\mu)$ are generic and spherical. The following theorem extends the above results to general spherical $\Pi(\chi_s)$ and $\pi(\mu)$.

Theorem 4.11. *For all choices of χ and μ , the following identity holds:*

$$(4.21) \quad \mathcal{Z}_{\nu}(s, f_{\chi}, f_{\mu}, \psi_{\ell, \kappa}) = \frac{L(s + \frac{1}{2}, \tau \times \pi)}{L(s+1, \tau \times \sigma) L(2s+1, \tau, \text{Asai} \otimes \xi_{E/F}^m)} \langle f_{\mu}, f_{\sigma} \rangle_{\sigma} \zeta(\chi_{\tau}, 1),$$

where $\langle f_{\mu}, f_{\sigma} \rangle_{\sigma}$ and $\zeta(\chi_{\tau}, 1)$ are independent of s . Moreover, if we normalize W_j so that $W_j(f_{\chi})(e) = 1$, then

$$(4.22) \quad \mathcal{Z}_{\nu}(s, f_{\chi}, f_{\mu}, \psi_{\ell, \kappa}) = \frac{L(s + \frac{1}{2}, \tau \times \pi)}{L(s+1, \tau \times \sigma) L(2s+1, \tau, \text{Asai} \otimes \xi_{E/F}^m)} \langle f_{\mu}, f_{\sigma} \rangle_{\sigma},$$

Proof. This proof is similar to Theorem 5.2 in [GPSR97]. By Proposition 4.5, it is sufficient to show that

$$\mathcal{Z}_\nu(s, f_\chi * \Phi_0, f_\mu, \psi_{\ell, \kappa}) = P(\chi, \mu)$$

holds for all choices of χ and μ .

Define

$$f^* \left(\begin{pmatrix} g & & \\ & h & \\ & & g^* \end{pmatrix} u \epsilon_{0,1} \eta n k \right) = \tau(g) f_\tau | \det g |_E^s \delta_{P_j}^{\frac{1}{2}} \sigma(h) f_\sigma,$$

where $g \in \text{Res}_{E/F}(\text{GL}_j)$, $h \in G_{n-j}$, $u \in U_j$, $n \in U_\ell(\mathfrak{o})$ and $k \in K_H$. Recall that f_τ and f_σ are the unramified spherical vectors in τ and σ . In addition, we assume that $\text{supp}(f^*) = P_{j\epsilon_{0,1}\eta} R_\ell(\mathfrak{o})$. Then f^* is in Λ and $\text{supp}(f^*) \subseteq G_{n-j} K_H$. Since $\mathcal{J}(f^*)(e) = W_j(f^*)(\epsilon_{0,1}\eta) = \zeta(\chi_\tau, 1) f_\sigma$, we obtain

$$\mathcal{Z}(s, f^*, f_\mu, \psi_{\ell, \kappa}) = \zeta(\chi_\tau, 1) \langle f_\mu, f_\sigma \rangle_\sigma.$$

Define

$$f^\sharp = f_\chi * \Phi_0 - \frac{d(\chi_\tau)}{L(s+1, \tau_\nu \times \sigma_\nu) L(2s+1, \tau_\nu, \text{Asai} \otimes \xi_{E/F}^m)} f^*.$$

By (4.19) and (4.20), if χ and μ are in general position and s is in a dense open set, then

$$\mathcal{Z}_\nu(s, f^\sharp, f_\mu, \psi_{\ell, \kappa}) = 0.$$

By the same argument, one can extend the *Density Principle* in Appendix IV to [GPSR97, Section 5] to the unitary group case, which implies that $\mathcal{J}(f^\sharp)(g) = 0$ for all choices of χ , μ and s . Therefore, we obtain the following identity

$$\mathcal{Z}_\nu(s, f_\chi * \Phi_0, f_\mu, \psi_{\ell, \kappa}) = \mathcal{Z}_\nu(s, f^*, f_\mu, \psi_{\ell, \kappa}) = P(\chi, \mu),$$

for all choices of χ , μ and s . \square

This completes the proof of Theorem 3.9, which is the key result for unramified local zeta integrals. With Theorems 3.8 and 4.11, we have the following main global result of this paper for $j = \ell + 1$. In this case, (H_{n-j+1}, G_{n-j}) is a spherical pair, and the Bessel period $\mathcal{P}^{\psi_{\beta^{-1}, y-\kappa}}(\varphi_\pi, \varphi_\sigma)$ reduces to a spherical Bessel period.

Theorem 4.12 (Main). *Assume that $j = \ell + 1$. Let $E(\phi_{\tau \otimes \sigma}, s)$ be the Eisenstein series on $G_n(\mathbb{A})$ as in (2.15) and let π be an irreducible cuspidal automorphic representation of $H_{n-\ell}(\mathbb{A})$. Assume that the real*

part of s , $\operatorname{Re}(s)$, is large, and that π and σ have a non-zero spherical Bessel period. Then the global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ is eulerian, and is equal to

$$c_{\pi, \sigma} \mathcal{Z}_S(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0}) \frac{L^S(s + \frac{1}{2}, \pi \times \tau)}{L^S(s + 1, \sigma \times \tau) L^S(2s + 1, \tau, \text{Asai} \otimes \xi_{E/F}^m)},$$

where $c_{\pi, \sigma}$ is a constant depending on the Bessel period of π and σ and on other normalization constants, but independent of s , and

$$\mathcal{Z}_S(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0}) = \prod_{v \in S} \mathcal{Z}_v(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$$

is the finite product of ramified local zeta integrals.

There is a standard method to prove from this global identity that the partial L -functions $L^S(s + \frac{1}{2}, \pi \times \tau)$ has meromorphic continuation to the whole complex plane. It is more important to develop the local theory which extends the partial L -function to the complete L -function in this setting and hence to prove the functional equation and other analytic properties of the complete L -functions of this type. This is our on-going project and will be reported in our future work.

5. FINAL REMARK

We remark that the proof of Theorem works for replacing the single variable s by a multi-variable (s_1, s_2, \dots, s_r) , and hence the resulting global zeta integral represents the following product of tensor product L -functions

$$L^S(s_1, \pi \times \tau_1) L^S(s_2, \pi \times \tau_2) \cdots L^S(s_r, \pi \times \tau_r).$$

We will come back to this in our future work.

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SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455, USA

E-mail address: dhjiang@math.umn.edu

DEPARTMENT OF MATHEMATICS,, BOSTON COLLEGE, CHESTNUT HILL, MA 02467, USA

E-mail address: lei.zhang.2@bc.edu